Microscopic models of quantum-jump superoperators

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We discuss the quantum-jump operation in an open system and show that jump superoperators related to a system under measurement can be derived from the interaction of that system with a quantum measurement apparatus. We give two examples for the interaction of a monochromatic electromagnetic field in a cavity (the system) with two-level atoms and with a harmonic oscillator (representing two different kinds of detectors). We show that the derived quantum-jump superoperators have a "nonlinear" form $J\rho=\gamma\,\mathrm{diag}[F(\hat{n})a\rho a^{\dagger}F(\hat{n})]$, where the concrete form of the function $F(\hat{n})$ depends on assumptions made about the interaction between the system and detector. Under certain conditions the asymptotical power-law dependence $F(\hat{n})=(\hat{n}+1)^{-\beta}$ is obtained. A continuous transition to the standard Srinivas-Davies form of the quantum-jump superoperator (corresponding to $\beta=0$) is shown.

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I. INTRODUCTION

In the theory of continuous photodetection and continuous measurements the (one-count) quantum-jump superoperator (QJS) is an essential part of the formalism [1–12], since it accounts for the loss of one photon from the electromagnetic field (EM) and corresponding photoelectron detection and counting within the measurement apparatus (MA). One of the main equations in this theory is the evolution equation of the field's density operator ρ_t, or master equation, which reads in the simplest variant as

$$\frac{d\rho_t}{dt}=\frac{1}{i\hbar}[H_0,\rho_t]-\frac{\gamma}{2}(O^{\dagger}O\rho_t+\rho_t O^{\dagger}O-2O\rho_t O^{\dagger}), \tag{1}$$

where H_0 is the EM field Hamiltonian, γ is the field-MA coupling constant, and O is some lowering operator, representing the loss of a single photon from the field to the environment, which may be detected and counted by a duly constructed experimental setup. Defining the effective non-Hermitian Hamiltonian as [13–16]

$$H_{\mathrm{eff}}=H_0-i\frac{\gamma}{2}O^{\dagger}O, \tag{2}$$

Eq. (1) can be written as (we set here $\hbar=1$)

$$\frac{d\rho_t}{dt}=-i(H_{\mathrm{eff}}\rho_t-\rho_t H_{\mathrm{eff}}^{\dagger})+\gamma O\rho_t O^{\dagger}, \tag{3}$$

whose formal solution is (see, for example, [1,17])

$$\rho_t=\sum_{k=0}^{\infty}\int_0^t dt_k\int_0^{t_k}dt_{k-1}\cdots\int_0^{t_2}dt_1 e^{L(t-t_k)}J\times e^{L(t_k-t_{k-1})}J\cdots Je^{Lt_1}\rho_0, \tag{4}$$

where

$$L\rho_0=-i[H_{\mathrm{eff}}\rho_0-\rho_0 H_{\mathrm{eff}}^{\dagger}],$$

ρ_0 being the density operator for the field state at $t=0$. The no-count superoperator $\exp[L(t_k-t_{k-1})]$ evolves the initial state ρ_0 from time t_{k-1} to the latter time t_k without taking out any photon from the field; it represents the field monitoring by a MA. The QJS $J\bullet=\gamma O\bullet O^{\dagger}$ is an operation which takes out instantaneously one photon from the field. Actually, $\mathrm{Tr}[J\rho_0]$ is the rate of photodetection [18].

The explicit form of the QJS is not predetermined. In the phenomenological photon counting theory developed by Srinivas and Davies (SD) [18] the QJS was introduced *ad hoc* as

$$J_{SD}\bullet=\gamma_{SD}a\bullet a^{\dagger}. \tag{5}$$

Later, Ben-Aryeh and Brif [19] and de Oliveira *et al.* [20] considered QJS's of the form

$$J_E\bullet=\gamma_E E_-\bullet E_+, \tag{6}$$

where

$$E_-=(a^{\dagger}a+1)^{-1/2}a \quad \text{and} \quad E_+=E_-^{\dagger} \tag{7}$$

are the exponential phase operators of Susskind and Glogower [21,22]. These "nonlinear" operators allow one to remove some inconsistencies of the SD theory noticed by its authors.

However, the QJS (6) was introduced in [19,20] also *ad hoc*. Therefore it is desirable to have not only a phenomenological theory, but also some *microscopic models*, which could justify the phenomenological schemes. The simplest example of such a model was considered for the first time in

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[23], where the QJS of Srinivas and Davies was derived under the assumption of highly efficient detection. The two fundamental assumptions of that model were (a) a infinitesimally small interaction time between the field and MA and (b) the presence of only a few photons in the field mode. Only under these conditions one can use a simple perturbative approach and arrive at the mathematical expression for the QJS, which is independent of the details of interaction between the MA and EM.

If the condition (a) or (b) is not fulfilled, the QJS should depend on many factors, such as, for example, the kind of interaction between the field and MA, the interaction strength, and the time T of the interaction. Moreover, it should be emphasized that the instant t_j at which the quantum jump occurs cannot be determined exactly—it can happen randomly at any moment within T. Making different assumptions concerning the moment of the "quantum jump," one can obtain different formal expressions for the QJS. In [24] we have proposed a simple heuristic model for obtaining the nonlinear QJS of the form

$$J\bullet = \gamma F(a^\dagger a)a\bullet a^\dagger F(a^\dagger a). \tag{8}$$

In this connection, the aim of the present paper is to provide a more rigorous derivation of QJS's, using a more sophisticated model that takes into account dissipation effects due to the "macroscopic part" of the MA. Our approach is based on the hypothesis that the transition probability must be averaged over the interaction time T, during which a photon can be gobbled by the detector at any time in the interval (0, T). Considering two different models of MA's—a two-level atom and a harmonic oscillator interacting with a single-mode EM field—we shall demonstrate that different kinds of interaction result in quite different QJS's.

The plan of the paper is as follows. In Sec. II we derive the QJS using the modified Jaynes-Cummings model (with account of damping due to the spontaneous decay of the excited state) and calculating the time average of the transition operator. In Sec. III we apply the same scheme to the model of two coupled oscillators, showing explicitly how the variation of the relative strength of coupling constants results in the change of the function $F(a^\dagger a)$ in Eq. (8). Sec. IV contains a summary and conclusions.

II. MODEL OF TWO-LEVEL ATOM DETECTOR

Let us consider first the model, which is a straightforward generalization of the one studied in [23]. The role of the "system" is played by a single mode of the electromagnetic field, while the "detector" of the MA (subsystem constituting the MA that actually interacts with the EM field) consists of a single two-level atom. The Hamiltonian for the total system is chosen in the standard form of the Jaynes-Cummings model [25],

$$H_0 = \frac{1}{2}\omega_0\sigma_0 + \omega\hat{n} + ga\sigma_+ + g^* a^\dagger\sigma_-, \tag{9}$$

where the Pauli pseudospin operators σ_0 and $\sigma_\pm$ correspond to the atom ($\sigma_+ = |e\rangle\langle g|$, $\sigma_- = |g\rangle\langle e|$, and $\sigma_0 = |e\rangle\langle e| - |g\rangle\langle g|$) and one considers that there were chosen two levels of the atom (the ground state $|g\rangle$ with frequency ω_g and the excited state $|e\rangle$ with frequency $\omega_e = \omega_g + \omega_0$); a, $a^\dagger$, and $\hat{n} = a^\dagger a$ are the lowering, rising, and number operators, respectively, of the EM field. Since the coupling between the field and atom is weak, we assume that $\omega \gg |g|$. Until now, the detector can absorb and emit photons back into the EM field, since the detector is not coupled to some macroscopic device that irreversibly absorbs the photons.

Therefore, we have to take into consideration that the detector is coupled to the "macroscopic part" (MP) of the MA (e.g., phototube and associated electronics). Hence the detector suffers dissipative effects responsible for the spontaneous decay of the excited level of the detector (in this case of the atom). And it is precisely this physical process that represents a photodetection—the excited level of the detector decays, emitting a photoelectron into the MP of the MA, which is amplified by appropriate electronics and is seen as a macroscopic electrical current inside the MP of the MA. We can take into account this dissipation effects by describing the whole photodetection process, including the spontaneous decay, by the master equation

$$\frac{d\rho_t}{dt} + i(H_{\text{eff}}\rho_t - \rho_t H_{\text{eff}}^\dagger) = 2\lambda\sigma_-\rho_t\sigma_+, \tag{10}$$

which is the special case of Eq. (3), where $O = \sigma_-$, $O^\dagger = \sigma_+$, $H_{\text{eff}} = H_0 - i\lambda\sigma_+\sigma_-$, and 2λ is the coupling of the excited level of the atom (detector) to the MP of the MA (here we make a reasonable assumption that λ has the same order of magnitude as $|g|$). The "sink" term

$$R\bullet = 2\lambda\sigma_-\bullet\sigma_+ \tag{11}$$

represents the $|e\rangle \to |g\rangle$ transition within the detector (the atomic decay process in this case). If $\lambda = 0$, then the detector interacts with the EM field, but photoelectrons are not emitted (thus no counts happen), because the absorbed photons are emitted back to the field and then reabsorbed at a later time, periodically, analogously to the Rabi oscillations.

In the following, we shall use the quantum trajectories approach [1]. The effective Hamiltonian (2) becomes

$$H_{\text{eff}} = H - i\lambda\sigma_+\sigma_- = \frac{1}{2}(\omega_0 - i\lambda)\sigma_0 + \omega\hat{n} + ga\sigma_+ + g^* a^\dagger\sigma_- - i\lambda/2 \tag{12}$$

[where we have used $\sigma_+\sigma_- = (1+\sigma_0)/2$] and the evolution of the system between two spontaneous decays is given by the no-count superoperator

$$\mathcal{D}_t\rho_0 = U(t)\rho_0 U^\dagger(t), \quad U(t) = \exp(-iH_{\text{eff}}t). \tag{13}$$

After a standard algebraic manipulation [25,26] we obtain the following explicit form of the *nonunitary* evolution operator $U(t)$:

$$U(t) = \exp[-i\omega(\sigma_0/2 + \hat{n})t]\left\{\frac{1}{2}\left[C_{\hat{n}+1}(t) - i\frac{\delta}{|g|}S_{\hat{n}+1}(t)\right]\right.$$
$$\times(1+\sigma_0) - i\frac{g}{|g|}S_{\hat{n}+1}(t)a\sigma_+ - i\frac{g^*}{|g|}a^\dagger S_{\hat{n}+1}(t)\sigma_-$$
$$\left. + \frac{1}{2}\left[C_{\hat{n}}(t) + i\frac{\delta}{|g|}S_{\hat{n}}(t)\right](1-\sigma_0)\right\}, \qquad (14)$$

where

$$C_{\hat{n}}(t) \equiv \cos(|g|B_{\hat{n}}t), \quad S_{\hat{n}}(t) \equiv \sin(|g|B_{\hat{n}}t)/B_{\hat{n}}, \qquad (15)$$

$$B_{\hat{n}} = \sqrt{\hat{n} + (\delta/|g|)^2}, \quad \delta = \frac{1}{2}(\omega_0 - \omega - i\lambda) \qquad (16)$$

(note that the parameter δ is complex and $\hat{n}$ is an operator).

Assuming that the field state is $\rho_0 = \rho_F \otimes |g\rangle\langle g|$ at time $t=0$ or, analogously, the last photoemission occurred at $t=0$, the probability that the *next* photoelectron emission will occur within the time interval $[t, t+\Delta t)$ is given by [1,18,27]

$$P(t) = \mathrm{Tr}_{F-D}[R\mathcal{D}_t\rho_0]\Delta t \qquad (17)$$

(the subscripts F and D are a reminder that the trace operation is on *field* and *detector* spaces, respectively), where Δt is the time resolution of the MA. Tracing out first over the detector variables, the probability density for the next photoemission to occur at time t will be [27]

$$p(t) = \lim_{\Delta t \to 0}\frac{P(t)}{\Delta t} = \mathrm{Tr}_F[\Xi(t)\rho_F], \qquad (18)$$

where the time-dependent *transition superoperator*

$$\Xi(t)\bullet = 2\lambda\Gamma(t)\bullet\Gamma^\dagger(t), \qquad (19)$$

acting on the EM field, stands for the photoelectron emission into the MP of the MA (i.e., the actual photodetection). Once again, the probability for detecting a photoelectron in $[t, t+\Delta t)$ is $P(t) = \mathrm{Tr}_F[\Xi(t)\rho_F]\Delta t$ (from now on we omit the subscript and write $\rho_F \equiv \rho$ for the field operator). In Eq. (19), $\Gamma(t)$ is the time-dependent *transition operator*

$$\Gamma(t) = \langle e|U(t)|g\rangle, \qquad (20)$$

which takes out a single photon from the field state. Substituting Eq. (14) into Eq. (20) we can write $\Gamma(t)$ as

$$\Gamma(t) = -i\frac{g}{|g|}\exp(-\lambda t/2 - i\omega\hat{n}t)S_{\hat{n}+1}(t)a, \qquad (21)$$

so the time-dependent transition superoperator (19) becomes

$$\Xi(t)\rho = 2\lambda e^{-\lambda t}e^{-i\omega\hat{n}t}S_{\hat{n}+1}(t)a\rho a^\dagger S^\dagger_{\hat{n}+1}(t)e^{i\omega\hat{n}t}. \qquad (22)$$

In the resonant case, $\omega_0 = \omega$, we have

$$B_{\hat{n}} = \sqrt{\hat{n} - \chi^2}, \quad \chi \equiv \lambda/(2|g|). \qquad (23)$$

If the interaction time Δt is small and the number of photons in the field is not very high, in the sense that the condition

$$|g|\Delta t\sqrt{n+1} \ll 1 \qquad (24)$$

is fulfilled for all eigenvalues of $\hat{n}$, for which the probabilities $p_n = \langle n|\rho|n\rangle$ are important, then one can replace the operator $\sin(B_{\hat{n}+1}|g|\Delta t)$ in Eq. (15) simply by $B_{\hat{n}+1}|g|\Delta t$ and arrive at the QJS,

$$J\rho = e^{-i\omega\hat{n}\Delta t}[2\lambda(|g|\Delta t)^2 a\rho a^\dagger]e^{i\omega\hat{n}\Delta t}, \qquad (25)$$

which has *almost* the Srinivas-Davies form (5), with the coupling constant

$$\gamma_{\mathrm{SD}} = 2\lambda(|g|\Delta t)^2. \qquad (26)$$

Taking $2\lambda = (\Delta t)^{-1}$ we obtain the same coupling constant γ_{SD} as in [23], but this assumption is not the only possible. Note that superoperator (25) contains the factors $\exp(\pm i\omega\hat{n}\Delta t)$, which can be essentially different from the unit operator even under condition (24), for two reasons: (1) the condition $|g|\Delta t \ll 1$ does not imply $\omega\Delta t \ll 1$, because $\omega \gg |g|$; (2) the condition (24) contains the square root of n, whereas the eigenvalues of $\exp(\pm i\omega\hat{n}\Delta t)$ depend on the number n itself, which is much greater than $\sqrt{n}$ if $n \gg 1$. Consequently, even the simplest microscopic model gives rise to a QJS, which is, strictly speaking, different from the SD jump superoperator, coinciding with the former only for the diagonal elements $|n\rangle\langle n|$ of the density matrix in the Fock basis.

If condition (24) is not satisfied, we propose that the QJS can be defined by *averaging* the transition superoperator (22) over the interaction time T, because the exact instant within (0, T) at which the photodetection occurs in each run is unknown, so a reasonable hypothesis is that these events happen randomly with uniform probability distribution

$$J_T\rho = \frac{1}{T}\int_0^T dt\ \Xi(t)\rho. \qquad (27)$$

Writing the field density operator as

$$\rho = \sum_{m,n=0}^{\infty}\rho_{mn}|m\rangle\langle n|, \qquad (28)$$

we have

$$J_T\rho = \sum_{m,n=1}^{\infty}\rho_{mn}\sqrt{mn}f_{mn}|m-1\rangle\langle n-1|, \qquad (29)$$

where

$$f_{mn} = \frac{2\lambda}{T}\int_0^T e^{i\omega t(n-m)-\lambda t}S_m(t)S_n(t)dt. \qquad (30)$$

It is natural to suppose that the product λT is big enough, so that the photodetection can happen with high probability. Mathematically, it means that we assume that $\exp(-\lambda T) \ll 1$. If $\lambda \ll \omega$ (this is also a natural assumption), then the off-diagonal coefficients f_{mn} with $m \neq n$ are very small due to fast oscillations of the integrand in Eq. (27), so they can be neglected [a rough estimation gives for these terms the order of magnitude $O(\lambda/\omega)$, compared with the diagonal coefficients f_{nn}]. Consequently, the microscopic model leads to the nonlinear *diagonal* QJS of the form

$$J\rho = \gamma\,\mathrm{diag}[F(\hat{n})a\rho a^{\dagger}F(\hat{n})], \tag{31}$$

where $\mathrm{diag}(\hat{A})$ means the diagonal part of the operator $\hat{A}$ in the Fock basis. The function $F(n)$ can be restored from the coefficients f_{nn} (apart the constant factor which can be included in the coefficient γ) as

$$F(n) = \sqrt{f_{n+1,n+1}}. \tag{32}$$

Under the condition $\exp(-\lambda T) \ll 1$, the upper limit of integration in Eq. (27) can be extended formally to infinity, with exponentially small error. Then, taking into account the definition of the function $S_n(t)$, Eq. (15), we arrive at integrals of the form

$$\int_0^{\infty} dt\, e^{-\lambda t} \times \begin{cases} \sin^2(\mu t)/\mu^2 & \text{for } \chi < 1, \\ t^2 & \text{for } \chi = 1, \\ \sinh^2(\mu t)/\mu^2 & \text{for } \chi > 1, \end{cases}$$

which can be calculated exactly (see, e.g., Eqs. 3.893.2 and 3.541.1 from [28]). The final result does not depend on λ or χ (and it is the same for either $\chi<1$ or $\chi>1$):

$$f_{nn} = (nT)^{-1}. \tag{33}$$

Thus we obtain the QJS

$$J_T\rho = \gamma_T \sum_{n=1}^{\infty} \rho_{nn}|n-1\rangle\langle n-1| = \gamma_T \mathrm{diag}(E_-\rho E_+), \tag{34}$$

where $\gamma_T = T^{-1}$ and the operators E_- and E_+ are defined by Eq. (7). Notice that, in principle, γ_T *is different* from γ_{SD}. Moreover, the superoperator (34) derived from the microscopic model turns out to be *different* from the phenomenological QJS (6) studied in [20,24]. The difference is that J_T has no off-diagonal matrix elements, while J_E has. We see that the microscopic model concerned (which can be justified in the case of big number of photons in the field mode) predicts that each photocount not only diminishes the number of photons in the mode exactly by 1, but also destroys off-diagonal elements, which means the total decoherence of the field due to the interaction with MA.

Note, however, that the formula (33) holds under the assumption that the upper limit of integration in Eq. (30) can be extended to the infinity. But this cannot be done if the parameter χ is very big. Indeed, for $\chi>1$ and $\lambda T \gg 1$, the integrand in Eq. (30) at $t=T$ is proportional to $\exp[-\lambda T(1-\sqrt{1-n/\chi^2})]$, so it is not small when $n/\chi^2 \ll 1$. Calculating the integral in the finite limits under the conditions $n/\chi^2 \ll 1$ and $\lambda T \gg 1$, we obtain the approximate formula

$$f_{nn} = (Tn)^{-1}\{1 - \exp[-\lambda T n/(2\chi^2)]\}, \tag{35}$$

which shows that f_{nn} does not depend on n if $\lambda T n/(2\chi^2) \ll 1$. Thus we see how the QJS (34) can be continuously transformed to the SD jump superoperator (5), when the number n changes from big to relatively small values. It should be emphasized, nonetheless, that the off-diagonal coefficients f_{mn} remain small even in this limit. Their magnitude approaches that of the diagonal coefficients only in the case of $\lambda \sim \omega$, which does not seem to be very physical.

III. MODEL OF HARMONIC OSCILLATOR DETECTOR

Now let us consider another model, where the role of the detector is played by a harmonic oscillator interacting with one EM field mode. This is a simplified version of the model proposed by Mollow [29] (for its applications in other areas see, e.g., [30] and references therein). In the rotating-wave approximation (whose validity was studied, e.g., in Ref. [31]) the Hamiltonian is

$$H = \omega_a a^{\dagger}a + \omega_b b^{\dagger}b + gab^{\dagger} + g^* a^{\dagger}b, \tag{36}$$

where the mode b assumes the role of the detector and the mode a corresponds to the EM field (ω_b and ω_a are the corresponding frequencies and g is the detector-field coupling constant). In the following we shall repeat the same procedures we did in the Sec. II. The dissipation effects due to the macroscopic part of the MA, associated with the mode b, can be taken into account by means of a master equation of the form

$$\frac{d\rho}{dt} + i[H_{\mathrm{eff}}\rho - \rho H_{\mathrm{eff}}^{\dagger}] = 2\lambda b\rho b^{\dagger}, \tag{37}$$

with the effective Hamiltonian

$$H_{\mathrm{eff}} = H - i\lambda b^{\dagger}b = (\omega_b - i\lambda)b^{\dagger}b + \omega_a a^{\dagger}a + gba^{\dagger} + g^* b^{\dagger}a. \tag{38}$$

The evolution operator $U(t)=\exp(-iH_{\mathrm{eff}}t)$ for the *quadratic* Hamiltonian (38) can be calculated by means of several different approaches [32]. Here we use the algebraic approach [7,33–35], since Hamiltonian (38) is a linear combination of the generators of algebra $su(1,1)$:

$$K_+ \equiv b^{\dagger}a, \quad K_- \equiv -ba^{\dagger}, \quad K_0 \equiv (b^{\dagger}b - a^{\dagger}a)/2,$$

$$[K_0, K_{\pm}] = \pm K_{\pm}, \quad [K_-, K_+] = 2K_0.$$

The evolution operator can be factorized as

$$U(t) = e^{-i\Omega t N} e^{A(t)K_+} e^{B(t)K_0} e^{C(t)K_-}, \tag{39}$$

where

$$N \equiv (b^{\dagger}b + a^{\dagger}a)/2, \quad \Omega \equiv \omega_b + \omega_a - i\lambda.$$

The time-dependent coefficients are

$$A(t) = -\frac{ig^*\sin(\eta t)}{\eta\Upsilon(t)}, \quad C(t) = \frac{ig\sin(\eta t)}{\eta\Upsilon(t)}, \tag{40}$$

$$B(t) = -2\ln\Upsilon(t), \tag{41}$$

with

$$\Upsilon(t) = \cos(\eta t) + i[\omega_{ba}/(2\eta)]\sin(\eta t), \tag{42}$$

$$\omega_{ba} \equiv \omega_b - \omega_a - i\lambda, \quad \eta \equiv (|g|^2 + \omega_{ba}^2/4)^{1/2}. \tag{43}$$

Assuming that the detector is in resonance with the EM field's mode one gets $\omega_{ba} = -i\lambda$ and

$$\Upsilon(t) = \cos(\eta_0 t) + [\lambda/(2\eta_0)]\sin(\eta_0 t), \tag{44}$$

$$\eta_0 = (|g|^2 - \lambda^2/4)^{1/2}. \tag{45}$$

If, initially, the detector oscillator is in the ground state $|0_b\rangle$, the time-dependent transition operator, corresponding to the absorption of one photon from the EM field, defined in Eq. (20), is

$$\Gamma(t) = \langle 1_b | U(t) | 0_b \rangle = A(t) \exp\left[-\frac{1}{2}[i\Omega t + B(t)](a^\dagger a + 1) \right] a \tag{46}$$

and the transition superoperator becomes

$$\Xi(t)\rho = 2\lambda |A(t)|^2 \exp\left[-\frac{1}{2}[i\Omega t + B(t)](a^\dagger a + 1) \right] \times a\rho a^\dagger \exp\left[\frac{1}{2}[i\Omega^* t - B^*(t)](a^\dagger a + 1) \right]. \tag{47}$$

For "small" $t=\Delta t$ and few photons in the cavity, the QJS (25) is recovered. Considering, instead, the time-averaged QJS, one has Eqs. (27)–(29). For $\chi = \lambda/(2|g|) < 1$ (when the parameter η_0 is real) one can represent the coefficients f_{mn} as (we consider the resonance case with $\omega_a = \omega_b = \omega$)

$$f_{mn} = \frac{4\chi}{T(1-\chi^2)^{3/2}} \int_0^Z dz [\cos(z) + \xi \sin(z)]^{m+n-2} \times \sin^2(z) \exp[i\bar{\omega} z(n-m) - \xi z(m+n)], \tag{48}$$

where

$$\xi = \frac{\chi}{\sqrt{1-\chi^2}}, \quad \bar{\omega} = \frac{\omega}{|g|\sqrt{1-\chi^2}}, \quad Z = \frac{\lambda T}{2\xi}. \tag{49}$$

Since the parameter $\bar{\omega}$ is big, the off-diagonal coefficients f_{mn} with $n \neq m$ are very small due to the strongly oscillating factor $\exp[i\bar{\omega} t(n-m)]$. Consequently, they can be neglected in the first approximation, and we arrive again at the diagonal QJS of the form (31).

We notice that the exact analytical expression for the integral in Eq. (48) is so complicated (even if $m=n$) that it is difficult to use it. For example, in the limit $\chi \to 1$, Eq. (48) can be reduced to the form

$$f_{nn} = \frac{4}{T} \int_0^{\lambda T/2} dy\, y^2 (1+y)^{2n-2} \exp(-2ny). \tag{50}$$

Replacing the upper limit by infinity, we recognize the integral representation of the Tricomi confluent hypergeometric function $\Psi(a;c;z)$ [36]. Thus we have [neglecting small corrections of the order of $\exp(-\lambda T)$]

$$f_{nn} = \frac{8}{T} \Psi(3; 2n+2; 2n). \tag{51}$$

Although the Ψ Function in the right-hand side of Eq. (51) can be rewritten in terms of the associated Laguerre polynomials [36] as

$$\Psi(3; 2n+2; 2n) = \frac{(2n)!}{2(2n)^{1+2n}} L_{2n-2}^{(-1-2n)}(2n), \tag{52}$$

neither Eq. (51) nor Eq. (52) help us to understand the behavior of the coefficient f_{nn} as function of n. Therefore it is worth trying to find simple approximate formulas for the integral in Eq. (48).

If $\chi \ll 1$, then also $\xi \ll 1$, so we can neglect the term $\xi \sin(z)$ in the integrand of Eq. (48) and the function $\sin^2(z) \times [\cos(z)]^{2n-2}$ can be replaced by its average value taken over the period 2π of fast (in the scale determined by the characteristic time ξ^{-1}) oscillations. After simple algebra we obtain (replacing the upper limit of integration Z by infinity)

$$f_{nn} = \frac{4(2n-2)!}{T(2^n n!)^2}, \quad \chi \ll 1. \tag{53}$$

Using Stirling's formula $n! \approx \sqrt{2\pi n}(n/e)^n$, we can write, for $n \gg 1$,

$$f_{nn} \approx (T\sqrt{\pi n^5})^{-1}. \tag{54}$$

This function corresponds to the QJS (31) with

$$F(\hat{n}) = F_5(\hat{n}) \equiv (\hat{n}+1)^{-5/4}, \quad \gamma = \gamma_5 \equiv (T\sqrt{\pi})^{-1}. \tag{55}$$

Thus, differently from the case of a two-level detector, in the simplest version of the oscillator detector model the lowering operator contains the factor $(\hat{n}+1)^{-5/4}$, instead of $(\hat{n}+1)^{-1/2}$ as in the "E model" (6) or simply $\hat{1}$ as in the SD model (5).

The case $\chi \ll 1$ is not very realistic from the practical point of view, since it corresponds to the detector with very low efficiency. However, we can calculate the integral (48) with arbitrary ξ approximately, assuming that $n \gg 1$ and using the *method of steepest descent*. Rewriting the integrand as $\exp[G(z)]$, one can easily verify that the points of maxima of the function

$$G(z) = 2\ln[\sin(z)] + 2(n-1)\ln[\cos(z) + \xi \sin(z)] - 2\xi nz$$

are given by the formula $z_k = \pm z_0 + k\pi$, where

$$z_0 = \tan^{-1}(\mu), \quad \mu = (\xi^2 n + n - 1)^{-1/2}, \tag{56}$$

$k=0, 1, 2, \ldots$ for the plus sign and $k=1, 2, \ldots$ for the minus sign. One can verify that

$$\exp[G(z_k)] = \frac{\mu^2(1+\xi\mu)^{2n-2}}{(1+\mu^2)^n} \exp(-2z_0\xi n - 2\xi\pi nk). \tag{57}$$

The second derivatives of the function $G(z)$ at the points of maxima do not depend on k:

$$G''(z_k) = -\frac{4n(\xi^2+1)}{1+\xi\mu}. \tag{58}$$

Using Eqs. (57) and (58) and performing summation over k we find (taking $Z=\infty$)

$$f_{nn} = \frac{\chi\sqrt{8\pi}(1+\xi\mu)^{2n-3/2}\exp(-2z_0\xi n)}{T\sqrt{n}(n+\chi^2-1)(1+\mu^2)^n}\coth(\xi n\pi). \quad (59)$$

Although application of the steepest-descent method can be justified for $n \gg 1$, formula (59) seems to be a good approximation for $n \sim 1$, too. For example, for $n=1$ (when $\mu=\xi^{-1}$) it yields

$$Tf_{11} \approx 4\chi\sqrt{\pi}\coth(\pi\xi)\exp[-2\xi\tan^{-1}(\xi^{-1})], \quad (60)$$

and the numerical values of Eq. (60) in the whole interval $0<\chi<1$ are not very far from the exact value $Tf_{11}=1$, which holds independently of χ, as far as the upper limit of integration in Eq. (48) can be extended to the infinity.

For $n \gg 1$ (when $\mu \ll 1$), Eq. (59) can be simplified as

$$f_{nn}(\chi) \approx \frac{\chi\sqrt{8\pi}}{eT}n^{-3/2}\coth\left(\frac{\chi n\pi}{\sqrt{1-\chi^2}}\right), \quad \chi \leqslant 1. \quad (61)$$

For $\chi \ll 1$ the function (61) assumes the form (54), with slightly different coefficient $\gamma' = (eT)^{-1}\sqrt{8/\pi} \approx 1.04\gamma_5$.

For $\chi > 1$ (when parameter η_0 is imaginary) we have, instead of Eq. (48), the integral (considering diagonal coefficients only)

$$f_{nn} = \frac{4\chi}{T(\chi^2-1)^{3/2}}\int_0^Y dz[\cosh(z)+\zeta\sinh(z)]^{2n-2} \times \sinh^2(z)\exp(-2n\zeta z), \quad (62)$$

where

$$\zeta = \chi/\sqrt{\chi^2-1}, \quad Y = \lambda T/(2\zeta). \quad (63)$$

Applying again the steepest-descent method, we have now the only point of maximum

$$z_{\max} = \tanh^{-1}(\nu), \quad \nu = [(\zeta^2-1)n+1]^{-1/2}. \quad (64)$$

Taking into account the value of the second derivative of the logarithm of integrand at this point,

$$G''(z_{\max}) = -\frac{4n(\zeta^2-1)}{1+\zeta\nu}, \quad \zeta\nu = \frac{\chi}{\sqrt{n+\chi^2-1}}, \quad (65)$$

we obtain

$$f_{nn} = \frac{\chi\sqrt{8\pi}(1+\zeta\nu)^{2n-3/2}(1-\nu)^{n(\zeta-1)}}{T\sqrt{n}(n+\chi^2-1)(1+\nu)^{n(\zeta+1)}}. \quad (66)$$

One can check that the limit of formula (66) at $\chi \to 1$ coincides with the analogous limit of formula (59), so the transition through the point $\chi=1$ is continuous.

The asymptotical form of Eq. (66) for $n \gg \chi^2$ is the same as Eq. (61), except for the last factor:

$$f_{nn}(\chi) \approx \frac{\chi\sqrt{8\pi}}{eT}n^{-3/2}, \quad \chi \geqslant 1, \quad (67)$$

Applying the steepest-descent method to the integral (50) (for $n \gg 1$), we obtain the same result (67) with $\chi=1$. Thus for $\chi \sim 1$ (not too small and not too large) we obtain the QJS in the form (31) with

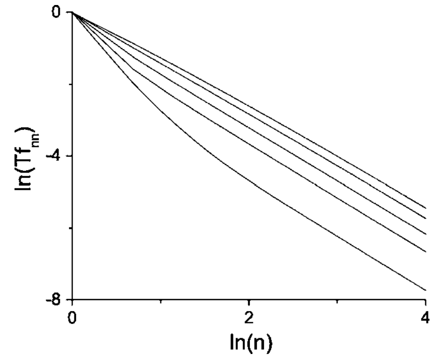

FIG. 1. Dependence of diagonal coefficients f_{nn} on the number n, obtained by numerical integration of Eqs. (48) and (62) with the fixed value $\lambda T=10$, for small and moderate values of the parameter χ (from below): $\chi=0.1$, 0.3, 0.5, 0.8, 1.1.

$$F(\hat{n}) = F_3(\hat{n}) \equiv (\hat{n}+1)^{-3/4}, \quad \gamma = \gamma_3 \equiv \frac{\chi\sqrt{8\pi}}{eT}. \quad (68)$$

For very large values of parameter χ (exceeding $\sqrt{n}$) the steepest-descent method cannot be used, because the second derivative of the logarithm of integrand, given by Eq. (65), becomes small and because the coordinate $z_{\max}$, determined by Eq. (64), tends to infinity, while the upper limit Y of integration in Eq. (62) tends to the fixed value $\lambda T/2$. For $\chi \gg 1$, Eq. (68) holds for the values of n satisfying approximately the inequality $n > n_* \sim 4\chi^2\exp(-\lambda T)$. If $n < n_*$, then it can be shown that Eq. (62) leads to the same approximate formula (35) as in the model of a two-level detector, so the SD superoperator (however, without off-diagonal elements) is restored for relatively not very large values of n.

In Figs. 1 and 2 we show the dependence of diagonal coefficients f_{nn} on the number n for different values of the parameter χ, obtained by numerical integration of Eqs. (48) and (62) for a fixed value of the parameter $\lambda T=10$; in Fig. 3, we compare them with the approximate analytical formulas (59) and (66). We see that the coincidence is rather satisfactory for large values of n, although there are some differ-

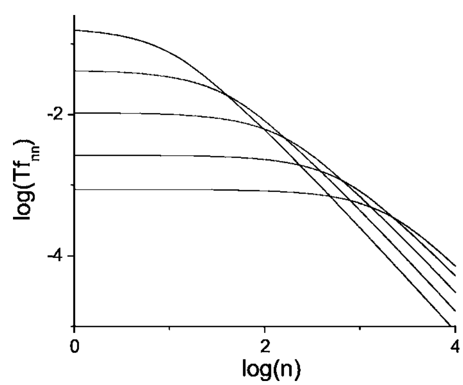

FIG. 2. Dependence of diagonal coefficients f_{nn} on the number n, obtained by numerical integration of Eq. (62) with the fixed value $\lambda T=10$, for large values of the parameter χ (from above): $\chi=5$, 10, 20, 40, 70. Notice the appearance of plateaus corresponding to the SD model for initial values of n; for large n they are transformed in curves with the slope given by a power-law dependence.

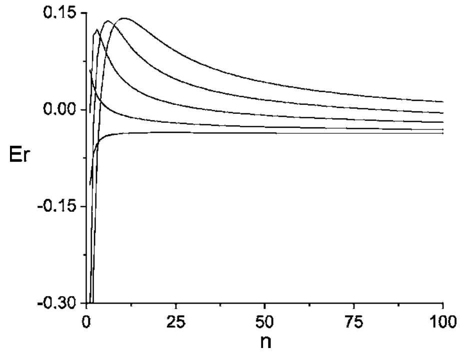

FIG. 3. Comparison of numerical integration of Eqs. (48) and (62) with the approximate analytical formulas (59) and (66) for $\chi=0.5$, 1.1, 2, 3, 4 (from below). We defined relative error by $Er=(f_{nn}^{\mathrm{num}}-f_{nn}^{\mathrm{anal}})/f_{nn}^{\mathrm{num}}$.

ences for $n\sim 1$. We also see in Fig. 2 that the increase of the parameter χ results in the appearance of the SD plateau for small values of n, which goes into a slope corresponding to the power-law dependence for large values of n. The height of the plateaus diminishes as χ^{-2} in accordance with Eq. (35), because large values of χ correspond (for fixed values of λ and T) to small coupling coefficient $|g|^2$ between the field and MA and, consequently, low probability of photocount.

IV. CONCLUSIONS

Here we presented two microscopic models for deducting QJS's. In the first one we supposed that the detector behaves like a two-level atom and in the second as a harmonic oscillator. The main difference between our models and previous ones is that we take into account the dissipative effects that arise when one couples the actual detector to the phototube. This scheme includes the spontaneous decay of the detector with originated photoelectron emission inside the phototube, which is amplified and viewed as macroscopic electric current. Using quantum trajectories approach we deduced general time-dependent transition superoperator, responsible for taking out a single photon from the field. Since it depends explicitly on interaction time, we proposed two distinct schemes for obtaining time independent QJS's from it. In the first case we assumed that the interaction time is small and that there are few photons in the cavity; in this situation we recovered the QJS proposed by Srinivas and Davies in both detector models. As a second scheme, we calculated time-averaged QJS on the time interval during which a photon is certainly absorbed; as the result, we obtained different nonlinear QJS's for the two-level atom model and the model of harmonic oscillator. In particular, we have shown that for quantum states with the predominant contribution of Fock components with large values of n, the QJS has the *nonlinear* form (31) with the power-law asymptotic function $F(\hat{n})=(\hat{n}+1)^{-\beta}$. However, the concrete value of the exponent β is model dependent. For the two-level atom model we obtained $\beta=1/2$, whereas in the model of harmonic oscillator the values $\beta=5/4$ and $\beta=3/4$ were found, depending on the ratio between the spontaneous decay frequency of the excited state and the effective frequency of coupling between the detector and field mode. Also, we have demonstrated how the simple Srinivas-Davies QJS arises in the case of states with a small number of photons. Another important result we obtained is that the QJS's, when applied to density matrix nondiagonal elements, are null on average in both models due to the strong oscillations of the free-field terms.

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