Accuracy of a teleported cavity-field state

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We present a calculation of the fidelity of a cavity-field state teleported by means of a scheme that requires only two high-$Q$ cavities. Based on current experimental capabilities, we demonstrate the feasibility of our scheme if the mean photon number of the cavity field is on the order of unity, allowing a reasonably accurate teleportation.

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Quantum nonlocality has recently become the cornerstone of a set of striking proposals, which open the way to new quantum technologies: bit commitment [1], teleportation [2], computation [3], and communication [4]. Inspired by these theoretical advances, experimentalists have developed techniques for the realization of teleportation [5] and to demonstrate quantum logic operations [6]. However, the realization of all these proposals faces a crucial problem, intrinsic to the nature of entanglement: the decoherence of quantum states subjected to the action of the environment. This drastic process, transforming superpositions of quantum states into statistical mixtures, has been discussed by a number of authors [7,13] and, recently, observed experimentally [14]. Since the coherence decays with a lifetime proportional to the inverse of the system excitation number [7], it becomes indispensable to estimate, from a given superposition, the fidelity of a resulting process. Recently, Braunstein and Kimble [8] have computed the fidelity of a teleported “Schrödinger cat”-like state (SCLS) generated by parametric down-conversion as a running wave. Here, however, we focus on the specific problems of quantum-state engineering and teleportation in cavity quantum electrodynamics (CQED). In this realm, a SCLS teleportation protocol has recently been proposed, requiring three high-$Q$ cavities [9]. In the present paper, we advance computation of the fidelity of a trapped SCLS, teleported through a scheme based on only two high-$Q$ cavities, whose feasibility we also analyze. The advantage of adopting our two-cavity scheme for teleporting a SCLS, as opposed to its alternative one in the literature [9], is twofold: it minimizes field dissipation through cavity damping mechanisms and also provides topological simplification of the experimental setup.

Previously presented two-cavity schemes [10,11] have not envisaged teleportation of mesoscopic field states. In this work, the quantum channel employing a mixed atomic-field state, prepared through the interaction between a two-level atom and a cavity field, is more suitable to mesoscopic-state teleportation. Teleportation of $N(>2)$-dimensional states has also been proposed in the CQED domain [12].

The teleportation apparatus is depicted in Fig. 1. The SCLS engineering consists of a two-level Rydberg atom $A$ that crosses a Ramsey-type arrangement, i.e., a high-$Q$ micromaser cavity $C_1$ located between two Ramsey zones $R_1$ and $R_2$. After interacting with this arrangement, the atom is counted by detector $D_A$, projecting the SCLS in $C_1$. Subsequently, this SCLS is teleported from Alice’s cavity $C_1$ to Bob’s identical cavity $C_2$. The quantum channel required for teleportation is here played by another two-level Rydberg atom $B$, entangled to the receptor field in $C_2$. Additional Rydberg atoms, Ramsey zones, and ionization chambers are used as needed.

The two-level atoms comprehend the circular Rydberg states [1] and [2], with principal quantum numbers 50 and 51, respectively. Each micromaser cavity is initially prepared in a single-mode coherent state $|\alpha_0\rangle$ by a monochromatic source. Classical microwave fields are injected into the cavities and the amplitudes of these fields can be adjusted by varying the injection time.

Here we take into account the errors introduced only by the cavity dissipation mechanism. The efficiency of the ionization chambers is considered to be unity, and two out of three atoms are able to travel the distance of the whole setup without decaying. In fact, since a Rydberg-atom excited state has a lifetime on the order of $10^{-2}$ s, the probability of staying in this state is 0.67 for an experiment duration of about $4 \times 10^{-3}$ s.

Under these assumptions the Hamiltonian including the required dispersive interaction between an atom and the dissipating cavity field is described, through the assumption of a linear coupling, as

$$H = \hbar \omega a^\dagger a + \frac{\hbar \omega_0}{2} \sigma_z + \sum_k \hbar \omega_k b_k^\dagger b_k$$

$$+ \sum_k \hbar (\lambda_k a^\dagger b_k + \lambda_k^* a b_k^\dagger) + \hbar \chi a^\dagger a \sigma_z,$$  (1)

FIG. 1. Experimental setup for engineering and teleporting the SCLS.
where $\sigma_z=|2\rangle\langle 2|-|1\rangle\langle 1|$, $a$ and $a^\dagger$ are, respectively, the creation and annihilation operators for the cavity mode of frequency $\omega$, whereas $b_k$ and $b_k^\dagger$ are the analogous operators for the $k$th bath oscillator mode, whose corresponding frequency and coupling constant write $\omega_0$ and $\lambda_k$, respectively. The frequency of the cavity mode lies between the atomic energy levels, separated by $\hbar\omega_0$, in a way that the detuning $\delta=[\omega-\omega_0]$ is large enough that only virtual transitions occur between the states $|1\rangle$ and $|2\rangle$. We assume that $\chi=0$ when the atom is outside the cavity. The atom-field coupling parameter is $\chi=\Omega^2/\delta$ inside the cavity, where $\Omega$ is the Rabi frequency. The last term in Eq. (1) is valid under the assumption that $\Omega^2 n<\delta^2+\gamma^2$, where $n$ is a characteristic photon number and $\gamma$ is the spontaneous-emission rate [15]. For simplicity, we suppose that the atom-field coupling is turned on (off) suddenly at the instant the atom enters (leaves) the cavity region.

The Schrödinger state vector associated with Hamiltonian (1) can be written using

$$|\Psi(t)\rangle=e^{i\omega_0 t/2}|1\rangle|\Phi_1(t)\rangle+e^{-i\omega_0 t/2}|2\rangle|\Phi_2(t)\rangle,$$

where

$$|\Phi_1(t)\rangle=f(d^2/\alpha,\pi)\{d^2 \beta_k, \pi\} A(\alpha,\{\beta_k\},t)|\alpha,\{\beta_k\}\rangle,$$

for $\ell=1,2$, the complex quantities $\alpha$ and $\beta_k$ stand for the eigenvalues of $a$ and $b_k$, respectively, and $A(\alpha,\{\beta_k\},t)$ are the expansion coefficients for $|\Phi_1(t)\rangle$ in the basis of coherent-state products, $\{|\alpha,\{\beta_k\}\rangle\}$. Using the orthogonality of the atomic states and Eqs. (1) and (2) we obtain the uncoupled time-dependent Schrödinger equations:

$$i\hbar \frac{d}{dt}|\Phi_\ell(t)\rangle=|\mathcal{H}_\ell|\Phi_\ell(t)\rangle,$$

$$\mathcal{H}_\ell=\hbar \omega_0 a^\dagger a+\sum_k \hbar \omega_0 b_k^\dagger b_k+\sum_k \hbar (\lambda_k a^\dagger b_k^\dagger+\lambda_k^* a b_k),$$

with $\omega_0=[\omega+(1)^\ell \chi]$. Note that the problem has been reduced to that of the dissipation of a cavity field whose frequency $\omega$ is shifted by $-\chi$ ($+\chi$) when interacting with the atomic state $|1\rangle$ ($|2\rangle$). To solve Eq. (3), we assume zero bath temperature, which is an excellent approximation for, in current experiments, the cavity environment is cooled to 0.6 K by a $^3$He-$^4$He refrigerator, making blackbody radiation negligible [14]. Moreover, in the microwave regime the states of the environmental oscillators can be approximated by the vacuum states, so that, initially, for a given cavity, $|\Phi_\ell(0)\rangle=|\alpha_0,\{\beta_0\}\rangle$. Here, we invoke the remarkable property, following from the Heisenberg equations of motion [7], that at absolute zero both the system and the environment remain coherent, despite exchanging excitation, in such a way that

$$|\Phi_\ell(t)\rangle=|\alpha_\ell(t),\{\beta_{\ell,0}(t)\}\rangle,$$

where the coherent-state amplitudes satisfy the time evolution equations

$$\dot{\alpha}_\ell(t)=-i\omega_0 \alpha_\ell(t)-i\sum_k \lambda_k \beta_{\ell,k}(t),$$

$$\dot{\beta}_{\ell,k}(t)=-i\omega_k \beta_{\ell,k}(t)-i\lambda_k^* \alpha_\ell(t).$$

Ideal process. Let us consider the ideal process, i.e., $\lambda_k=0$ in Hamiltonian (1). The SCLS to be teleported is prepared in $C_1$ where the coherent state $|\alpha_0\rangle$, has been previously injected. This is done by sending atom $A$ across $C_1$. The atom is assumed to be velocity selected such that $\chi \tau=\pi/2$, where $\tau$ is the atom-field interaction time. After being laser excited to the state $|2\rangle_A$ and rotated in $R_1$ to an arbitrary superposition $c_1|1\rangle_A+c_2|2\rangle_A$ at $t=0$, atom $A$ undergoes a dispersive interaction with the field in $C_1$ and a $\pi/2$ pulse in $R_2$. The process is accomplished by detecting the atom at $D_A$, inducing the collapse of the cavity field to the SCLS

$$|\Psi^\ominus_B(t)\rangle=\sum_{i=1}^N \sigma_B|c_i\rangle|\alpha_1\rangle_{1,z} \pm |c_2\rangle|\alpha_2\rangle_{1,z},$$

where $\sigma_B=-(1)^i |\alpha_0\rangle, N_\sigma$ is a normalization factor, and the $+(-)$ sign occurs if $A$ is detected in state $|1\rangle_A$ ($|2\rangle_A$). Simultaneously to the preparation of the SCLS, the quantum channel is generated by sending atom $B$, assumed velocity selected exactly as atom $A$, across $C_2$. After undergoing a $\pi/2$ pulse in $R_3$ (leaving the zone at $t=0$) and a dispersive interaction in $C_2$ the atom plus cavity system ends up, at time $t$, in the entangled state

$$|\Psi(t)\rangle_{B,2}=\frac{1}{\sqrt{2}}(e^{i\omega_0 t/2}|1\rangle_B|\alpha_2\rangle_{2,z}+e^{-i\omega_0 t/2}|2\rangle_B|\alpha_1\rangle_{2,z}),$$

where the phases $e^{\pm i\omega_0 t/2}$ are a consequence of adopting the Schrödinger picture.

The teleportation process is achieved when atom $B$ undergoes a dispersive interaction with $C_1$ and a $\pi/2$ pulse in $R_4$, leading the whole system to the entangled state

$$|\Psi(t)\rangle_{B,1,2}=-\sum_{i=1}^N \frac{N_{\sigma}}{2}(e^{i\omega_0 t/2}|1\rangle_B[|\alpha_0\rangle_1|c_i\rangle_1|\alpha_1\rangle_2 \pm |c_2\rangle_2|\alpha_2\rangle_1]\times\pm|\alpha_0\rangle_1|c_1\rangle_1|\alpha_1\rangle_2 \pm |c_2\rangle_2|\alpha_2\rangle_1)\sum_{i=1}^N \pm|\alpha_0\rangle_1|c_2\rangle_2|\alpha_1\rangle_1 \pm |c_1\rangle_1|\alpha_2\rangle_2).$$

So, by detecting atom $B$ at $D_B$ and measuring the phase of the field in $C_1$, we project the field state in $C_2$ on one of the four possibilities described by Eq. (9). By injecting a reference field of known amplitude $\alpha_0$ in $C_1$, through its correspondent source, the field states $|\alpha_0\rangle_1$ and $|\alpha_0\rangle_1$ in Eq. (9) evolve respectively to the states $|0\rangle_1$ and $|2\alpha_0\rangle_1$. Such states can be easily distinguished by sending through $C_1$ a stream of ground-state two-level atoms that are made resonant with the cavity field [16]. Considering the outcome $|1\rangle_B|\alpha_0\rangle_1$, the SCLS teleported to $C_2$ is exactly the same as the original one. For the outcome $|1\rangle_B|\alpha_0\rangle_1$, one must now send an auxiliary atom, prepared in the excited state,
to interact dispersively with $C_2$, by selecting the velocity of this atom such that $\chi \tau = \pi$, the cavity mode in $C_2$ evolves exactly to the original SCLS. About the outcomes $|2\rangle_B - \alpha_{01}$, and $|2\rangle_B + \alpha_{01}$, one can still obtain, from the projected states in $C_2$, the original SCLS. However, this feat demands considerable effort, working against the fidelity of the teleportation process, as we will discuss elsewhere.

**Real process: preparation of the SCLS.** After the interaction between atom $A$ (prepared in the arbitrary state $c_1 |1\rangle_A + c_2 |2\rangle_A$) and the cavity field $|\alpha_{01}\rangle$, the state of the system is $c_1 (|1\rangle_A |\alpha_{11}(t)\rangle + c_2 |2\rangle_A |\alpha_{21}(t)\rangle)$, and the cavity field is $|\alpha_{01}\rangle$, the state of the system is $c_1 (|1\rangle_A |\alpha_{11}(t)\rangle + c_2 |2\rangle_A |\alpha_{21}(t)\rangle)$, where $c_1(t) = c_1 \exp[-(\frac{\Gamma}{2})\sqrt{2}D_A(t)]$. The field amplitudes $\alpha_{11}(t)$ and $\alpha_{21}(t)$ evolve according to Eqs. (6) during the time interval the atom spends inside the cavity from instant $t_0 > 0$ to $t_1$. The evolution outside the cavity obeys Eqs. (6) with $\chi = 0$. By detecting the atom at $D_A$ after undergoing a $\pi/2$ pulse in $R_2$, the resulting field-environment entangled state thus is written as

$$|\Psi_{FP}^+(t)\rangle = N_P |c_1 |\alpha_{11}(t)\rangle, |\beta_{1,1}(t)\rangle \rangle_1 + c_2 |\alpha_{21}(t)\rangle, |\beta_{2,1}(t)\rangle \rangle_2,$$  (10)

where $N_P$ is a normalization factor and the + (−) sign occurs if atom $A$ is detected in state $|1\rangle_A$ ($|2\rangle_A$).

The fidelity $F_P^+(t)$ of the prepared SCLS, i.e., the overlap between the state vector of Eq. (10) and the expected SCLS, $|\Psi_E^+(t)\rangle = N_P |c_1 |\alpha_{11}(t)\rangle, |\beta_{1,1}(t)\rangle \rangle_1 + c_2 |\alpha_{21}(t)\rangle, |\beta_{2,1}(t)\rangle \rangle_2$, is given by

$$F_P^+(t) = \langle |\Psi_E^+(t)| |\Psi_{FP}^+(t)\rangle \rangle^2$$

Here, $\Gamma$ is the relaxation constant of the cavities. The quantities $\alpha_{11}(t)$ are obtained from the expressions of $\alpha_{11}(t)$ evaluated for $\Gamma = 0$. For simplicity let $d_1 = c_1, d_2 = c_2$. Hence,

$$F_P^+(t) = N_P^2 \sum_{\mu, \nu, \mu', \nu'} d_{\mu, \nu}^* d_{\mu', \nu'} d_{\mu', \nu} \langle \alpha_{11}(t) | \alpha_{11}(t) \rangle \langle \beta_{1,1}(t) | \beta_{1,1}(t) \rangle,$$  (11)

where summations over Greek indices run from 1 to 2. To calculate the scalar products appearing in this equation, we use the exact results that follow from Eqs. (5) and (6).

**Teleportation of the SCLS.** Let us assume, without loss of generality, that atom $B$, velocity selected equally as atom $A$ and prepared in the state $(|1\rangle_B + |2\rangle_B) / \sqrt{2}$ by crossing $R_3$, reaches $C_2$ at time $t_1$ and leaves it at $t_2$. Thus, the required entangled quantum channel is given at $t > t_1$ as

$$|\Psi(t)\rangle_{B,2} = \frac{1}{\sqrt{2}} \sum e^{-\frac{i}{\sqrt{2}D_A(t)}} |\alpha_{11}(t)\rangle, |\beta_{1,1}(t)\rangle \rangle_2.$$

$$|\Psi(t)\rangle_{B,2} = \frac{1}{\sqrt{2}} \sum e^{-\frac{i}{\sqrt{2}D_A(t)}} |\alpha_{11}(t)\rangle, |\beta_{1,1}(t)\rangle \rangle_2.$$

(12)

Considering that atom $A$ is detected by $D_A$ at instant $t_3$, the tensor product $|\Psi_{FP}^+(t)\rangle_1 \otimes |\Psi(t)\rangle_{B,2}$ can be expressed at time $t > t_3$, before atom $B$ enters $C_1$, as

$$N_P \sum_{\ell, m} e^{\frac{-i}{\sqrt{2}D_A(t)}} |\alpha_{\ell,m}(t)\rangle, |\beta_{\ell,m}(t)\rangle \rangle_2 \times |\alpha_{\ell,m}(t)\rangle, |\beta_{\ell,m}(t)\rangle \rangle_2.$$

(13)

After atom $B$ interacts with $C_1$ during the time interval from $t_3 > t_3$ to $t_3$, a free evolution of the system follows until time $t_6$, when the atom undergoes a $\pi/2$ pulse in $R_4$ leading the entanglement (13), at instant $t > t_6$, to the state

$$N_P \sum_{\ell, m} e^{\frac{-i}{\sqrt{2}D_A(t)}} |\alpha_{\ell,m}(t)\rangle, |\beta_{\ell,m}(t)\rangle \rangle_2 \times |\alpha_{\ell,m}(t)\rangle, |\beta_{\ell,m}(t)\rangle \rangle_2,$$

(14)

with $\alpha_{\ell,m}(t) = \alpha_{\ell,m}(t) \times \alpha_{\ell,m}(t)$, where we have chosen $\chi \tau = \pi/2$, so that, from Eqs. (5) and (6),

$$\langle \alpha_{\ell,m}(t), |\beta_{\ell,m}(t)\rangle \rangle_2 = \alpha_0 e^{-\frac{i}{2}D_A(t)}.$$

(15)

and $\beta_{\ell,m}(t) > t'$ is the result of Eqs. (6) for $\beta_{\ell,m}(t)$ with initial condition given by $\beta_{\ell,m}(t)$.

To accomplish the teleportation we now measure, analogously to the corresponding lossless process, the state of atom $B$ in detector $D_B$ and the phase of the field in $C_1$. In about 25% of the trials the final teleported state will be exactly the original SCLS corresponding to the detection of $|1\rangle_B - \alpha(t)$:

$$|\Psi(t)\rangle_{B,2} = N_P e^{\frac{-i}{\sqrt{2}D_A(t)}} |\alpha_{\ell,m}(t)\rangle, |\beta_{\ell,m}(t)\rangle \rangle_2 \times |\alpha_{\ell,m}(t)\rangle, |\beta_{\ell,m}(t)\rangle \rangle_2,$$

(16)

whose fidelity $F_{TP}^+(t) = \langle |\Psi_E^+(t)| |\Psi_{TP}^+(t)\rangle \rangle^2$ is written

$$\begin{align*}
F(t) &= \frac{1}{\sqrt{2}} \sum_{\ell, m} e^{\frac{-i}{\sqrt{2}D_A(t)}} |\alpha_{\ell,m}(t)\rangle, |\beta_{\ell,m}(t)\rangle \rangle_2 \times |\alpha_{\ell,m}(t)\rangle, |\beta_{\ell,m}(t)\rangle \rangle_2.
\end{align*}$$

FIG. 2. Fidelity for both the prepared (solid line) and teleported (dashed line) SCLS, for $\alpha = 1/2, 1,$ and $3/2$. The SCLS decoherence times are $\tau_{\alpha} = 2\Gamma$, $\tau_1 = 1/2\Gamma$, and $\tau_{\alpha} = 2(9\Gamma)$, respectively (considering the photon lifetime $\Gamma^{-1} = 10^{-2}$ s).
\[ F_1^+(t) = |N_N^N_1|^2 \sum_{\mu,\nu,\mu',\nu'} d_{\mu} d_{\nu}^* d_{\mu'}^* d_{\nu'} \]
\[ \times \langle \{ \beta_{\mu',\nu}(t) \} | \{ \beta_{\mu,\nu}(t) \} \rangle^2 \]
\[ \times \langle a_{\mu}^{\Gamma=0}(t) | a_{\mu}(t) \rangle \langle a_{\mu}(t) | a_{\mu}^{\Gamma=0}(t) \rangle. \] (17)

In Fig. 2 we present the fidelity for both the prepared (11) and teleported SCLS’s (17), calculated with parameter values achievable using nowadays technology. For the photon lifetime we consider \( \Gamma^{-1} = 10^{-2} \) s, accounting for a cavity \( Q \) factor around \( 10^8 \) [14], whereas the atom-field coupling parameter is fixed so that \( \chi \tau = \pi/2 \). The effective atom-cavity interaction time is set to \( \tau = 30 \) \( \mu \)s by selecting atoms with velocity around 240 m/s, implying the typical value \( \chi/2\pi \approx 10 \) kHz [14]. Considering a Rabi frequency given by \( \Omega/2\pi \approx 25 \) kHz, and a detuning of \( \delta \approx 500 \) kHz (perfectly within the range of nowadays CQED technology), the condition for the dispersive atom-field coupling is satisfied, i.e., \( \Omega/\delta \leq 1 \).

A successful realization of the teleportation process is obtained for \( \alpha = 0.5 \). In this representative case both the fidelities are approximately unity, even at \( t = \Gamma^{-1} = 10^{-2} \) s, which is about one half of the SCLS decoherence time. For \( \alpha = 1 \), the fidelity of the prepared SCLS by the time the teleportation is concluded is definitely significant, \( \sim 0.75 \), showing that the teleported state has not been considerably degraded. Indeed, the teleportation is achieved before this SCLS decoherence time (\( 5 \times 10^{-3} \) s). For \( \alpha = 1.5 \), by the time the teleportation process is achieved, the prepared SCLS has already lost its initial quantum coherence, which occurs at about \( 2 \times 10^{-3} \) s. However, we stress that the fidelity of such a prepared SCLS starts with an appreciable value above 0.8. It is also apparent in Fig. 2 that, by the time the teleportation occurs, the fidelity of the prepared SCLS has already diminished appreciably. Thus, our result prevents any attempt to teleport a SCLS with \( \alpha > 1 \), regarding nowadays experimental capabilities. For \( \alpha = 1.5 \), despite the fact that the fidelity of the prepared SCLS is considerable, the fidelity of the teleported state drops to about 0.55. Hence, the present calculation shows why the experiments requiring just the preparation of a SCLS, as has been the case of Ref. [14], leads to results that fit very well the theoretical calculations.

We notice that we have indeed calculated a conditional fidelity, i.e., the one resulting from 25% of the trials concerning the expected measurement of the state of atom \( B \) in detector \( D_B \) and the phase of the field in \( C_1 \). However, one can still obtain, from whichever measurement results, the original SCLS, at the expense of a smaller fidelity, as we will discuss elsewhere.

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