A note on quantum oscillators coupled by nonresonant interaction: environment modifying dynamics

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We study the dynamics of a single fermionic system interacting non-resonantly with a single mode of a bosonic oscillator coupled to a reservoir. It is shown that the damping rates for both systems are modified when the action of their environments are frequency-dependent. The interest of such demand is discussed.

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Quantum decay processes of any unstable state is an early theme in physics, and it remains an up-to-date topic nowadays. Its importance comes from the fact that isolated systems - as usually required by quantum mechanics - are not realistic. In the real nature we actually deal with open systems [1], which are influenced by their surrounding world through the exchange of energy. Besides including studies on dissipation and amplification effects, the subject may also concern with a large variety of topics in physics. Among them one can cite: (i) the study of damping mechanisms causing atomic line-broadening [2] and field-mode linewidth [3, 4]; (ii) inhibition [5] and acceleration [6] of quantum decay rates processes concerning the Zeno and anti-Zeno effects [7], respectively; (iii) decoherence effects [8, 9], etc.

With respect to damping rate of a given system a very interesting result appears when the damping rate can be controlled to become small; this would correspond to rising the decoherence time of the state describing the system (e.g., the state describing a field-mode trapped inside a high-Q cavity [10, 11], or describing a mode of an atomic oscillator inside a magnetical-optical trap [12, 13]). Decoherence means loss of the “integrity” of a state, being intimately related to disappearance of superposition effects for macroscopic bodies. A clear understanding of decoherence in order to make it small is mandatory for constructing quantum computers and it is also important in studies concerning quantum information. Exploring the connection between damping rate $\gamma_s$ and decoherence effects we actually have that $\tau_{\text{dec}} = \tau_{\text{cav}} / < n >$ [10, 11], where $\tau_{\text{dec}}$ and $\tau_{\text{cav}} = 1/\gamma_s$ represent respectively the decoherence time and lifetime of the system whereas $< n >$ stands for its average excitation. Engineering decoherence became a hot-topic nowadays in view of its relevance for the developments of quantum computers and quantum information [14, 15]. In view of the previous connection between $\tau_{\text{dec}}$ and $\tau_{\text{cav}}$ and the additional fact that $\tau_{\text{cav}}$ is connected to the damping rate $\gamma_s$ of a cavity, in this paper we will point out a possible alternative way to control the decoherence effect upon the state describing a given system, via the control of its decay rate. We should mention that connection between damping rate and decoherence time is valid when the latter emerges from loss-reservoirs, as assumed in this report. This connection is meaningless when decoherence occurs without dissipation, e.g., when caused by the action of phase-reservoirs [16–18]. Yet, irrespective of our attention to decoherence effects, calculation of damping rates has its own interests, coming from other consequences of the modified system-dynamics [19, 20].

The damping rate $\gamma_s$ of a system $S$ realized as a quantum oscillator coupled to a reservoir is described by the Fermi’s golden rule [21, 28]

$$\gamma_s = 2\pi |\lambda(\omega_o)|^2 D(\omega_o), \quad (1)$$

where the frequency-dependent function $\lambda(\omega)$ is the coupling parameter system-reservoir and $D(\omega)$ is the density of states of the reservoir for system $S$; both functions $\lambda(\omega)$ and $D(\omega)$ taken at the resonance frequency $\omega_o$. The general case of a frequency-dependent density of states leads to interesting situations, since actions upon the environment can imply a modification of the $D(\omega)$ function, opening the possibility to modify the damping rate $\gamma_s$ of the system, and thus its dynamics. Here we study how the damping rate $\gamma_s$ is modified when the whole system concerns with a fermionic system interacting non-resonantly with a single mode of a bosonic oscillator; the latter being coupled to infinite modes of a bosonic reservoir. To be more specific, this model describes a two-level atom interacting dispersively with a single mode of a quantized electromagnetic field, trapped in a lossy QED-cavity studied in [22, 23]. In this scenario the whole system is represented by the Hamiltonian,

$$H = H_A + H_B + H_{A,B} + H_R + H_{A,R}, \quad (2)$$

with:

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where \( a^\dagger (a) \) is the creation (annihilation) bosonic operator for system \( A \) obeying \([a, a^\dagger] = 1\); \( b^\dagger_k (b_k) \) is the bosonic creation (annihilation) operator of the \( k \)-th oscillator of the reservoir, having frequency \( \omega_k \) and obeying the commutation relation \([b_j, b_k^\dagger] = \delta_{jk}; \omega_A (\omega_B) \) is the frequency of the single mode for system \( A \) (\( B \)); \( \lambda_k \) is a parameter that couples the system \( A \) to reservoir \( R \); \( \sigma^+ \) and \( \sigma^- \) are the Pauli spin operators obeying \([\sigma^+, \sigma^-] = \sigma_3 \), and \( \chi \) is a constant coupling system \( A \) to system \( B \). \( H_A \) and \( H_B \) are respectively the free Hamiltonian for the system \( A \) and the system \( B \) whereas \( H_{A,B} \) represents their non-resonant interaction; \( H_R \) stands for the free Hamiltonian describing the reservoir and \( H_{A,R} \) represents the interaction of the mode \( A \) with the reservoir in the usual rotating wave approximation \([25, 26]\). While Eq.(5) shows the single mode of the system \( A \) interacts with infinite modes of system \( B \) and Eq.(6) presents this single mode of the system \( B \) is coupled to infinite modes of system \( R \).

If \([1]_B \) and \([2]_B \) are the eigenstates of the pseudo fermionic operator \( \sigma_3 \), with eigenvalues \(-1 \) and \(+1\) respectively, then the Schrödinger state vector associated with the Hamiltonian (2) can be written as

\[
|\Psi (t)\rangle = e^{i\omega_B t/2}|1\rangle_B |\Phi_1 (t)\rangle + e^{-i\omega_B t/2}|2\rangle_B |\Phi_2 (t)\rangle,
\]

with

\[
|\Phi_\ell (t)\rangle = \int \frac{d^2 \alpha}{\pi} \int \frac{d^2 \beta_k}{\pi} A_\ell (\alpha, \{\beta_k\}, t)|\alpha, \{\beta_k\}\rangle,
\]

and \( \ell = 1, 2; \alpha \) and \( \beta_k \) are complex quantities standing for the eigenvalues of \( \alpha \) and \( \beta_k \) respectively, and the \( A_\ell (\alpha, \{\beta_k\}, t) \) are the expansion coefficients for \(|\Phi_\ell (t)\rangle\) in the basis of coherent-state products \([\prod_k |\alpha\rangle/\beta_k\rangle \equiv |\alpha, \{\beta_k\}\rangle\). Using the Eqs.(8),(9) and orthogonality of the states \([1]_B \) and \([2]_B \) we obtain the uncoupled \( \ell \)-dependent Schrödinger equations:

\[
\frac{d}{dt} |\Phi_\ell (t)\rangle = \mathcal{H}_\ell |\Phi_\ell (t)\rangle,
\]

with

\[
\mathcal{H}_\ell = \hbar \omega_A a^\dagger a + \sum_k \hbar \omega_k b_k^\dagger b_k + \sum_k \hbar (\lambda_k a^\dagger b_k + \lambda_k^* a b_k^\dagger),
\]

where \( \omega_\ell = (\omega_A + \chi) \) if initially the system \( B \) starts from \([1]_B \) and \( \omega_\ell = (\omega_A - \chi) \) when the system \( B \) initially starts from state \([2]_B \). So, our problem was reduced to the well known problem of a single bosonic mode interacting with an infinite collection of reservoir modes \([24]\). At this point it is straightforward to get the Fermi’s golden rule for both systems \( A \) and \( B \), either in Langevin approach or in Wigner-Weisskopf approximation \([21, 27, 28]\). For completeness, we delineate the steps leading to the Fermi’s golden rules in order to see where the dynamic is modified. Starting from the Hamiltonian given in Eq.(2) we obtain the Heisenberg equations of motion for operators \( a(t) \) and \( b_k(t) \):

\[
\frac{d}{dt} a(t) = -i \omega_A a(t) - \sum_k \lambda_k b_k(t),
\]

\[
\frac{d}{dt} b_k(t) = -i \omega_k b_k(t) - i\lambda_k^* a(t).
\]

We can formally integrate Eq.(13) and substitute the result in the Eq.(12) to obtain

\[
\frac{d}{dt} a(t) = -i \omega_A a(t) - \sum_k \lambda_k b_k(0) e^{-i \omega_k t} - \int_0^t \sum_k |\lambda_k|^2 a(t') e^{-i \omega_k (t-t')} dt'.
\]

If we now convert the sums over the reservoir modes to the integrals including the frequency-dependent density of states, i.e., \( \sum_k |\lambda_k|^2 \rightarrow \int S_R (\omega) d\omega \) with the resultant spectral density \( S_R (\omega) \equiv \langle |\lambda(\omega)|^2 D(\omega) \rangle \), then the last term in the Eq.(14) can be written as

\[
\int_0^t \sum_k |\lambda_k|^2 a(t') e^{-i \omega_k (t-t')} dt' \rightarrow \int_0^t a(t - \tau) d\tau \int_{-\omega_t}^{\omega_t} d\omega S_R (\omega_t + \tau) e^{-i \omega t'},
\]

where we have used \( \tau = t-t' \) and \( \omega = \omega_t + \tau \) by assuming that \( a(t) \) oscillates at frequency \( \omega_t \) at a first approximation. The \( \omega \)-integral gives, by the Wiener-Khintchine theorem, the autocorrelation function

\[
\int_{-\omega_t}^{\omega_t} d\omega S_R (\omega_t + \omega) e^{-i \omega t'} = 2\pi \Gamma (\tau),
\]

where \( \Gamma (\tau) \) is the “memory” function for the reservoir modes and clearly reduces to a Dirac’s delta function if \( S_R (\omega) \) is a constant, i.e., if the reservoir is a “white noise” type. Now, if \( t \) is large compared with the short range of \( \Gamma (\tau) \), we can remove \( a(t) \) out of the integrand in Eq.(15) and use the Fourier inversion of Eq.(16) to rewrite Eq.(14) as

\[
\frac{da(t)}{dt} = -(i \omega_t + \gamma_A) a(t) - \sum_k \lambda_k b_k(0) e^{-i \omega_k t},
\]

where \( \gamma_A = 2\pi S_R (\omega_A \pm \chi) \) is the damping rate for mode \( a \). Therefore, the dynamic of mode \( a \) is modified when
Next, let us now show how this result emerges from the Wigner-Weisskopf approximation. The Eq.(14) may be solved by Laplace transform. Firstly we rewrite the Eq.(14) in the form,

$$\frac{d}{dt} \tilde{a}(t) = -i \sum_k \lambda_k b_k(0) e^{-i(\omega_k - \omega)t} - \int_0^t \sum_k |\lambda_k|^2 \tilde{a}(t') e^{-i(\omega_k - \omega)(t-t')} dt',$$

where \(a(t) \rightarrow \tilde{a}(t)e^{-i\omega t}\). By taking the inverse Laplace transform of the Eq.(17), the last term can be written as

$$\sum_k |\lambda_k|^2 \int_0^t \tilde{a}(t') e^{-s(t-t')} dt' \rightarrow \sum_k \frac{|\lambda_k|^2}{s+i(\omega_k - \omega)},$$

where we have identified \(\tilde{a}(s) = \int_0^\infty \tilde{a}(t') e^{-st} dt'\) as the inverse Laplace of \(\tilde{a}(t)\). Thus, after Laplace transform the Eq.(17) furnishes

$$\tilde{a}(s) = \frac{\tilde{a}(0) - i \sum_k \lambda_k b_k(0)}{s + \sum_k |\lambda_k|^2/(s+i(\omega_k - \omega))}.$$ \hspace{1cm} \text{(19)}$$

In the Wigner-Weisskopf approximation \cite{21} we have

$$\sum_k \frac{|\lambda_k|^2}{s+i(\omega_k - \omega)} \rightarrow \lim_{s \rightarrow 0} \int d\omega S_R(\omega) \frac{1}{s+i(\omega - \omega)},$$

where \(S_R(\omega) \equiv |\lambda(\omega)|^2 D(\omega)\). Rewriting this equation as

$$\int d\omega S_R(\omega) \delta(\omega - \omega_i) + i \int d\omega \frac{S_R(\omega)}{(\omega - \omega_i)} \equiv \gamma_A + i \Delta \omega_i,$$

where \(\gamma_A = 2\pi S_R(\omega_i)\) is the damping rate for mode \(a\), and taking the Laplace inverse of Eq.(19) we finally obtain,

$$a(t) = a(0) e^{-i(\Delta \omega_i + \frac{\Delta}{2} t)} + \sum_k \tilde{b}_k(t) b_k(0),$$

where \(\tilde{b}_k(t)\) is a \(t\)-dependent function in which we have no interest for the moment. Here the relevant point is the damping rate \(\gamma_A = 2\pi S_R(\omega_A \pm \chi)\), which is clearly modified if \(\chi \neq 0\), provided \(S_R(\omega)\) is not a white-noise function.

Now, let us analyze the evolution of the system \(B\). We shall show that the dynamics of system \(B\) is also modified. To this end, note that from Eq.(22) we have the following mean value for mode \(a\):

$$\langle a^\dagger(0)a(\tau) \rangle = \pi_\omega(0) \exp \left[ -i(\omega_A \pm \chi) - \frac{\gamma_A}{2} \right] \tau,$$

where \(\pi_\omega(0)\) is the initial mean photon number and we have assumed \(\langle a^\dagger(0)b_i(0) \rangle = 0\). Next, the Wiener-Khintchine theorem \cite{27} yields the spectral density for the mode \(a\):

$$S_A(\omega) = \frac{1}{2\pi \pi_\omega(0)} \int_{-\infty}^{\infty} d\tau \langle a^\dagger(0)a(\tau) \rangle \exp(i\omega\tau) \equiv \frac{1}{\pi} \left[ \frac{\gamma_A}{2} \right]^2 + \left( \frac{\gamma_A}{2} \right)^2,$$

where the \(S_A(\omega)\) now represents the spectral density of mode \(a\). The foregoing Lorentzian distribution for mode \(a\) is the mode-density which system \(B\) “feels” while interacting with system \(A\). Thus, provided \(S_A(\omega)\) is not so smooth, there will be difference in \(S_A(\omega)\) if one takes different values for \(\chi\). This modified mode \(a\) reservoir will cause a change in the value of the Fermi golden rule for system \(B\), in the following way:

$$\gamma_B = 2\pi S_A(\omega_B) \rightarrow 2\pi S_A(\omega_B \pm \chi).$$

Finally, assume the reservoirs for systems \(A\) and \(B\) as frequency-dependent. Then \(\gamma_B\) will be influenced by two sources of changes: one stemming from the modification suffered by the Lorentzian distribution of mode \(a\), Eq.(24), when \(S_A(\omega_B)\) is taken at the shifted frequency \(\omega_B \rightarrow \omega_B \pm \chi\); the other coming from the modification \(\gamma_A = 2\pi S_R(\omega_A) \rightarrow 2\pi S_R(\omega_A \pm \chi)\) appearing in the Eq.(24).

In resume, we have studied two coupled systems interacting non-resonantly: one of them (\(A\)) composed by a single field mode, the other (\(B\)) realized as a two-level system. The first is also coupled to infinite modes of a bosonic reservoir in the usual rotating wave approximation. Using the Langlevin approach plus the Wigner-Weisskopf approximation it was shown that the damping rates \(\gamma_A\) and \(\gamma_B\) for systems \(A\) and \(B\) are modified, from \(\gamma_A = 2\pi S_R(\omega_A) \rightarrow 2\pi S_R(\omega_A \pm \chi)\), if their respective reservoirs are frequency-dependent. A striking result in this dynamics emerges from the possibility of controlling the damping rates, which would permit the control of decoherence-time through the mentioned connection between the quantities \(\gamma_A\) and \(\gamma_\text{dec}\). On the other hand, if the white-noise approximation is good enough, alternative ways for controlling the decoherence-time will depend on how to realize such frequency-dependent reservoirs. Although reservoirs engineering has recently been proposed \cite{14} and reported \cite{15} in the context of trapped ions, details on how to achieve this goal in practice, through the present model-system, is a point deserving future attention. As far as we know, our Eq.(25) is the first explicitly showing the change of damping rate due to the action of dispersive interaction with a reservoir. This result could be interesting in QED domain, where white noise reservoirs are usually assumed. Actually, our result gives an indicative of how to test the Markovian character of reservoirs by changing the system frequency \(\omega'\) and probing the sensivity of \(\gamma_B\) through \(S(\omega')\).
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