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Quantum photodetection distributions with 'nonlinear' quantum jump superoperators

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Abstract
We address the issue of the detection and counting of photons. We assume an electromagnetic field enclosed in an ideal cavity, which together with the detector constitute a closed system, so no photon is lost to the environment. Basing ourselves on a microscopic model consisting of a set of two-level atoms (the detector) interacting with the field, we derive a 'nonlinear' jump superoperator. We compare the count statistics calculated within our model with those obtained from previous models, such as the *coincidence probability density*, *two-count conditional probability density*, *waiting times* and the *second order correlation function*, for several field states.

Keywords: continuous photodetection, counting rate, Srinivas–Davies model, nonlinear raising and lowering operators, generalized Jaynes–Cummings model, intensity dependent interaction, exponential phase operators, E-model, master equation, thermal states, coherent states, Fock states

1. Introduction

The original proposal for describing the measurement mechanism in quantum mechanics is that of von Neumann, who conjectured that the measurement of an observable on a system entails its state reduction to one of its eigenstates, or in short, a sudden change of the system state by projection. However, on probing an electromagnetic (EM) field state through a photocount process, the photons are detected and counted one by one: a photon entering a photomultiplier tube provokes a burst of electrons (a photocurrent) which is viewed as originating from that single photon. It is then registered and counted. A sequence of bursts in a given time interval is associated to the photocount process. So, the determination of the field state is not achieved by an instantaneous projective measurement, but it takes some time t to count a sequence of photons, whose statistics gives information about the field state. The subject of photocounting has been addressed and detailed by many authors; in particular, see [1–6].

To describe the sequential photocount events in an ideal closed cavity more realistically, Srinivas and Davies (SD) [7] proposed an approach based on the concept of continuous measurement. Their scheme allows for calculating various statistical functions that can be compared with experimental outcomes, such as the probability of counting any number k of photons in a time interval t, and different coincidence probability densities. The SD model takes into account a 'back action' of the photodetector on the state of field and gives the conditioned field state, i.e., the field state just after the detector is turned off. Progress in understanding the physical meaning of the axiomatical SD model was achieved due to further studies [8–13] (for other references see [14, 15]).

However, as was pointed out from the very beginning [7], the SD theory shows some inconsistencies in calculating the coincidence probability densities for certain field states. This annoying feature is intimately related with the fundamental assumption that the statistical operator of the field $\hat{\rho}(t)$ immediately after a count of one photon 'jumps' to the state

$$\rho(t^+) = \frac{a\rho(t)a^\dagger}{\mathrm{Tr}[a^\dagger a\rho(t)]} = \frac{a\rho(t)a^\dagger}{\overline{n}(t)}, \tag{1}$$

where a and $a^\dagger$ are the field 'annihilation' and 'creation' operators, $[a, a^\dagger] = 1$, $\overline{n}(t)$ is the mean number of photons in the cavity just before the count, and t^+ stands for t plus an infinitesimal time after a photon is taken out of the field. Although the 'jump operator' $a\rho(t)a^\dagger$ in (1) seems to be quite natural, it is *unbounded*, explicitly violating the so-called 'fifth postulate' of the SD theory. Namely, this unboundedness results in nonexistence or strange behaviour of some probability distributions in the SD theory (see section 4).

One should remember, however, that equation (1) was simply *postulated* in [7], although some physical justifications were discussed [11]. Therefore, we suggest studying the possibility of replacing the operators a and $a^\dagger$ in (1) by some '*nonlinear*' operators[3]

$$A = F(\hat{n})a, \qquad A^\dagger = a^\dagger F(\hat{n}) \tag{2}$$

(where $F(\hat{n})$ is a real function of the photon number operator $\hat{n} \equiv a^\dagger a$), preserving all other ingredients of the SD theory. Nonlinear operators of the form (2) turned out to be very useful in different areas of quantum physics: from parastatistics and q-deformations [16, 17] to high energy physics [18] and physics of trapped ions [19–25] (where they were used for constructing *nonlinear coherent states*). In this connection, Ben-Aryeh and Brif did consider *ad hoc* nonlinear operators such as (2) for constructing quantum jump superoperators used in a discrete photodetection model [26]. We believe that such operators can be used in the continuous theory of photodetection, as well.[4]

Recently [27], we have shown that it is possible to get rid of inconsistencies, and still keep the structure of the SD theory, by means of replacing the operators a and $a^\dagger$ in equation (1) by the special case of (2)—the so-called 'exponential phase operators' [28–31]

$$E_- = (\hat{n}+1)^{-1/2}a, \qquad E_+ = a^\dagger(\hat{n}+1)^{-1/2}. \tag{3}$$

Such a change was motivated by the study of the role of the 'annihilation' operator a in quantum optics [32]. We have drawn attention to the fact (noticed also in [10, 13]) that a state $a|\psi\rangle$ is not always one whose mean number of photons is necessarily less than in $|\psi\rangle$; this occurs, partly, because of the presence of the 'weight' $\sqrt{n}$ in $a|n\rangle = \sqrt{n}|n-1\rangle$.

One of the goals of this paper is to show that nonlinear jump superoperators arise quite naturally from 'microscopical' models of photodetectors (generalizing the model used in [11]), if one makes one step forward, abandoning the short-time approximation and perturbation approach. This is done in section 2, where we derive a quantum jump superoperator from the structure of the time evolution operator of a system consisting of two-level atoms (considered as a 'detector') interacting with a single-mode EM field.

We show that the quantum jump superoperator based on operators (3) is, in certain sense, a limiting case of a family of possible nonlinear generalizations of the model based on the operator (1). For this reason, in the rest of the paper we concentrate on this special case (which we call the *E-model*, since it is based on the exponential phase operators $E_\pm$), comparing predictions of the new model with predictions of the SD theory (based on the operators a and $a^\dagger$) for several multitime probability density functions, which were not discussed in [27]. In section 3 we generalize the SD scheme to arbitrary nonlinear operators A instead of a and discuss some important differences between the SD model and the E-model. Then, in section 4 we study the following quantities:

(a) the *coincidence probability density* (CPD);
(b) the *two-count conditional probability density* (TCCP), which is the probability density of registering one count at time $t_2 = t_1+\tau$, if, with certainty, one count was registered at the earlier time t_1, *with any number of counts in between*;
(c) the *conditional probability count density* (CPC), i.e., the probability density for registering a count at time t_2, if, with certainty, one count was already registered at an earlier time t_1, *with no counts in between*;
(d) the *unconditional probability count density* (UPC), which has the following meaning: choosing any time t_1 *independently of a count having occurred or not* at that time, it is the probability density of registering the first arriving photon at time $t_2 = t_1 + \tau$.

Section 5 contains a summary, discussions and conclusions.

2. Nonlinear quantum jump operators

Let us consider a simple model Hamiltonian, describing the interaction between a two-level atom and an EM field,

$$\hat{H} = \tfrac{1}{2}\omega\sigma_0 + \omega\hat{n} + gf(\hat{n})a\sigma_+ + g^*a^\dagger f(\hat{n})\sigma_-, \tag{4}$$

where the Pauli operators σ_0 and $\sigma_\pm$ stand for a two-level atom. This is one of many possible nonlinear generalizations of the Jaynes–Cummings model [33]. For simplicity, we consider the case of exact resonance between the field mode and atom. Two kinds of measuring apparatus can be imagined.

(a) A sequence of atoms, prepared in the lower energy level (of the selected two levels), crosses the cavity one by one, with the possibility of absorbing one photon from the field ($f(\hat{n}) = 1$). If an atom carries one photon from the field, it will jump to the higher energy level, and at the exit of the cavity it undergoes a measurement that tells us about its energy level. If the atom is found in the higher level, the experimentalist knows that the cavity has lost one photon, otherwise, it continues to be in the same state, unknown to him/her; nevertheless, the information about the field state is being continuously refreshed.
(b) Another way to probe the field is by inserting a detector inside the cavity. In this case, the interaction is quite more complex, which suggests making the interaction term nonlinear, depending on the photon number operator through some function $f(\hat{n})$. In any case, following the well known techniques [34], after a straightforward

[3] In order to avoid confusion, we note that the word 'nonlinear' is understood in this paper in the sense 'nonlinear with respect to operators a and $a^\dagger$'; with respect to their action on states of the Hilbert space all operators are linear, of course.

[4] It is worth noting that a nonlinear operator (2) with $F(\hat{n}) = (\hat{n}+1)^{-1}$ appears in the scheme of reversible measurements considered in [12].

calculation we get an explicit form of the evolution operator $U(t) = \exp(-\mathrm{i}t\hat{H})$:

$$U(t) = \mathrm{e}^{-\mathrm{i}\omega(\sigma_0/2+\hat{n})t}\left\{\frac{1}{2}\left[\cos\left(|g|tf(\hat{n})\sqrt{\hat{n}+1}\right)\right.\right.$$
$$\left. + \cos\left(|g|tf(\hat{n}-1)\sqrt{\hat{n}}\right)\right]\mathbf{1}$$
$$+\frac{1}{2}\left[\cos\left(|g|tf(\hat{n})\sqrt{\hat{n}+1}\right)\right.$$
$$\left. - \cos\left(|g|tf(\hat{n}-1)\sqrt{\hat{n}}\right)\right]\sigma_0$$
$$-\frac{\mathrm{i}g}{|g|}\sin\left(|g|tf(\hat{n})\sqrt{\hat{n}+1}\right)\left(\frac{1}{\sqrt{\hat{n}+1}}a\right)\sigma_+$$
$$\left.-\frac{\mathrm{i}g^*}{|g|}\left(a^\dagger\frac{1}{\sqrt{\hat{n}+1}}\right)\sin\left(|g|tf(\hat{n})\sqrt{\hat{n}+1}\right)\sigma_-\right\}.$$

If, initially, the field is in an arbitrary state ρ_f and the atom is in the ground state $|0_\mathrm{a}\rangle\langle 0_\mathrm{a}|$, at a latter time t, the state of the atom–field system is

$$R(t) = U(t)(|0_\mathrm{a}\rangle\langle 0_\mathrm{a}| \otimes \rho_\mathrm{f})U^\dagger(t).$$

Since the 'jump' in the field state (associated with the click of the detector) occurs when the atom makes a transition from the initial ground state $|0_\mathrm{a}\rangle$ to the excited state $\langle 1_\mathrm{a}|$, the (*unnormalized*) statistical operator of the field becomes

$$\rho_{\mathrm{f},0\to 1}(t) = \langle 1_\mathrm{a}|R(t)|1_\mathrm{a}\rangle = \Gamma(t)\rho_\mathrm{f}\Gamma^\dagger(t), \qquad (5)$$

where

$$\Gamma(t) = \frac{\sin\left(|g|tf(\hat{n})\sqrt{\hat{n}+1}\right)}{\sqrt{\hat{n}+1}}a. \qquad (6)$$

Obviously, $\Gamma(t) \bullet \Gamma^\dagger(t)$ is a transition probability operator characterizing the jump operation for the specific interaction. The bullet stands for a density operator or, in particular, any elementary projector. We see that $\Gamma(t)$ has exactly the structure (2). It depends on the interaction time t, the strength of interaction between the field and atom $|g|$ *and the intensity of field* (through the photon number operator $\hat{n}$). If $f(\hat{n}) = 1$ and the interaction is very effective, such that the jump occurs immediately after the atom enters the cavity (for case (a)) or immediately as the photon impinges the detector surface (for case (b)), the time t can be quite small, so the sin function can be replaced by its argument. Only in this limiting case is the operator (6) reduced (after normalization) to the form (1) (cf the paper [11]).

However, the short time approximation *cannot be valid* for any intensity of the field, as is clearly seen from equation (6). Consequently, the jump operator *must* be nonlinear. Assuming this fact, one immediately removes the contradiction related to unboundedness of operator (1): a correct jump operator is, as a matter of fact, bounded, because it is *nonlinear*, and linearity arises only in the first approximation (as frequently happens), valid for small interaction times and low intensities of the field.

We propose to study the models based on the generic form (6) in another publication. Here we confine ourselves to its limiting case, suggested in [27] on 'heuristic' grounds; we call this the E-model, since it makes use of the nonlinear operators (3). It can be obtained in two different ways. In case (a), we can assume that a quantum jump occurs at time t when $\|\Gamma(t) \bullet \Gamma^\dagger(t)\|$ is maximum, or when each $\sin(\cdot) = 1$, so, $\Gamma(t) \to (\hat{n}+1)^{-1/2}a = E_-$. In case (b), since any real photodetector consists of a great number of atoms, we may suppose that the realistic jump superoperator can be obtained by averaging formula (5) over some distribution $P(|g|)$, which accounts for the effect of many atoms with different coupling constants with the field (and slightly different resonance frequencies in a generic case):

$$\overline{\Gamma(t)\rho_\mathrm{f}\Gamma^\dagger(t)} = \int P(|g|)\,\mathrm{d}|g|\,\frac{\sin\left(|g|tf(\hat{n})\sqrt{\hat{n}+1}\right)}{\sqrt{\hat{n}+1}}a\rho_\mathrm{f}a^\dagger$$
$$\times\frac{\sin\left(|g|tf(\hat{n})\sqrt{\hat{n}+1}\right)}{\sqrt{\hat{n}+1}}.$$

In the Fock basis, for $\rho_\mathrm{f} = \sum_{m,n} c_{m,n}|m\rangle\langle n|$, we have

$$\overline{\Gamma(t)\rho_\mathrm{f}\Gamma^\dagger(t)} = \int P(|g|)\,\mathrm{d}|g|\sum_{m,n=1}^{\infty} c_{m,n}|m-1\rangle\langle n-1|$$
$$\times\sin\left(|g|tf(m-1)\sqrt{m}\right)\sin\left(|g|tf(n-1)\sqrt{n}\right).$$

If the distribution $P(|g|)$ is sufficiently wide (and time t is not too small), all terms with $m \neq n$ are removed after averaging, while the integrals containing the terms with $m = n$ are reduced to constant values. Thus, for the two cases we arrive at the 'E-model', whose specific feature consists in using the following quantum jump superoperator instead of (1) (from now on we omit the subscript f for the field density operator):

$$\widetilde{\rho}(t^+) = \frac{\sum_{n=1}^{\infty} c_{n,n}|n-1\rangle\langle n-1|}{\sum_{n=1}^{\infty} c_{n,n}} = \frac{E_-\rho E_+}{\mathrm{Tr}[\Lambda\rho(t)]}. \qquad (7)$$

Here the operator $\Lambda \equiv E_+E_-$ replaces the photon number operator $\hat{n} \equiv a^\dagger a$ of the SD model. Obviously $[\Lambda, \hat{n}] = 0$. In addition, the following properties of the exponential phase operators are useful in calculations:

$$\left[E_-, E_+\right] = |0\rangle\langle 0| \equiv \Lambda_0 = 1 - \Lambda, \qquad \mathrm{e}^{\alpha\Lambda} = \Lambda_0 + \mathrm{e}^{\alpha}\Lambda,$$
$$E_-|n\rangle = (1-\delta_{n,0})|n-1\rangle, \qquad E_+|n\rangle = |n+1\rangle.$$

3. Generalization of the SD scheme

3.1. One-count events

The main ingredients in the SD theory stem from the use of two kinds of operations *acting continuously* on a field state. These operations are: the instantaneous *one-count event* (for a single photon count), represented by superoperator J, and the *no-count event* (zero photon count, field monitoring only) lasting for an arbitrary time interval τ, represented by the superoperator S_τ. So, in a time interval $[0, t)$ (when the detector–field interaction is on), all the one-count events constitute a zero measure set in the time line while all the no-counts have measure t. An obvious generalization of the SD one-count superoperator J is

$$J_A\rho = \gamma_A A\rho A^\dagger, \qquad (8)$$

where the operator A is given by equation (2) and the coefficient γ_A (it has inverse time dimension) is related to the detector efficiency. The physical meaning of the operator J_A is contained in the equation for the rate of photon counts [7, 10]. Namely, the probability $P(J_A)\,\mathrm{d}t$ that a one-count event occurs in the interval from t to $t+\mathrm{d}t$ is

$$P(J_A)\,\mathrm{d}t = \mathrm{Tr}[J_A\rho]\,\mathrm{d}t = \gamma_A\,\mathrm{Tr}[A^\dagger A\rho]\,\mathrm{d}t. \qquad (9)$$

Hereafter we shall use the symbol γ instead of γ_A in the case of the SD model and $\tilde{\gamma}$ for the E-model (having in mind that γ and $\tilde{\gamma}$ can be quite different). Moreover, it will be assumed everywhere that operators without a tilde correspond to the SD model, whereas a tilde over a quantity will mean that this quantity is calculated according to the E-model. Using this agreement, we have

$$P(J) = \gamma\overline{n}, \qquad P(\widetilde{J}) = \widetilde{\gamma}(1-p_0) = \widetilde{\gamma}\overline{\Lambda}, \qquad (10)$$

where $p_0 = |c_{0,0}|^2$ is the probability of presence of the vacuum component in the state of field.

Here the difference between the two models is clearly seen. The SD model is based on the assumption that the photocount rate is proportional to the mean number of photons $\overline{n}$, which seems quite natural for *low* field intensities (but can be untrue for high intensities). In contrast, in the E-model, the photocount rate is proportional to the probability that there are photons in the cavity $\overline{\Lambda}$, which has no direct relations with the mean number $\overline{n}$. Such a situation seems natural for *high* field intensities, when the count rate reaches some saturation. Hence, both models can be considered as two extreme limiting cases of a generic situation.

The mean numbers of photons in the state $\rho(t^+)$ immediately after a one-count event, predicted by the SD model and the E-model, are as follows:

$$\overline{n}(t^+) = \overline{n} + q, \qquad \widetilde{n}(t^+) = \frac{\overline{n}}{(1-p_0)} - 1, \qquad (11)$$

where q is Mandel's parameter

$$q = \left(\overline{\Delta n^2} - \overline{n}\right)\Big/\overline{n}, \qquad \overline{\Delta n^2} \equiv \overline{n^2} - \overline{n}^2. \qquad (12)$$

It is worth considering three examples of field states.

(a) For the *number state* $|m\rangle$ (with $m \neq 0$),

$$\overline{n}(t^+) = \widetilde{n}(t^+) = m - 1.$$

(b) For the *coherent state* $|\alpha\rangle$ (with $\alpha \neq 0$),

$$\overline{n}(t^+) = \overline{n}, \qquad \widetilde{n}(t^+) = \overline{n}/(1-\mathrm{e}^{-\overline{n}}) - 1.$$

(c) For the *thermal state*

$$\rho = \sum_{n=0}^{\infty} \frac{\overline{n}^n}{(1+\overline{n})^{n+1}} |n\rangle\langle n|, \qquad (13)$$

one obtains $\overline{n}(t^+) = 2\overline{n}$ [10, 13, 32], whereas $\widetilde{n}(t^+) = \overline{n}$, because of the special property $p_0 = 1/(\overline{n}+1)$.

3.2. No-count operators

Whenever the detector is monitoring the field and no counts are registered in the time interval $[0, t)$, the knowledge of the field is changed due to the continuous monitoring (continuous refreshing of information), and the initial statistical operator of the field ρ_0 is continuously changed to [7, 10]

$$\overline{\rho}(t) = \frac{S_t\rho_0}{\mathrm{Tr}(S_t\rho_0)} \qquad (14)$$

and the probability of not registering a single count within the time t is $P(0,t) = \mathrm{Tr}(S_t\rho_0)$. The SD superoperator S_t can be naturally generalized as

$$S_\tau\rho = \mathrm{e}^{Y_A\tau}\rho\,\mathrm{e}^{Y_A^\dagger\tau}, \qquad Y_A = -\mathrm{i}H_0 - \tfrac{1}{2}\gamma_A A^\dagger A, \qquad (15)$$

where $H_0 = \omega_0 a^\dagger a$ is the free field Hamiltonian. The evolution equation for the density matrix of a free field coupled to a detector can be written as

$$\mathrm{i}\frac{\mathrm{d}\rho}{\mathrm{d}t} = [H_0, \rho] + J_A\rho - \frac{1}{2}\gamma_A\left\{A^\dagger A, \rho\right\}, \qquad (16)$$

where the last term is introduced in order to preserve $\mathrm{Tr}\,\rho$ in time. This equation was solved for $A = a$ in [10, 15] and for $A = E_-$ in [27].

The SD model and the E-model predict the following no-count probabilities ($p_n \equiv \langle n|\rho_0|n\rangle$),

$$P(0,t) = \sum_{n=0}^{\infty} \mathrm{e}^{-n\gamma t} p_n, \qquad \widetilde{P}(0,t) = p_0 + (1-p_0)\,\mathrm{e}^{-\widetilde{\gamma}t}.$$

Although the two expressions go to the same limit as $t \to \infty$, $\lim_{t\to\infty}\widetilde{P}(0,t) = p_0$, at intermediate times they are different.

The mean number of photons in the absence of counts, $\overline{n}_t \equiv \mathrm{Tr}(\bar{\rho}_t\hat{n})$, also behaves differently in the two models. The SD model yields

$$\overline{n}_t = \frac{\sum_{n=1}^{\infty} n\mathrm{e}^{-n\gamma t} p_n}{\sum_{n=0}^{\infty} \mathrm{e}^{-n\gamma t} p_n}. \qquad (17)$$

In particular, for $\gamma t \ll 1$, $\overline{n}_t \approx \overline{n} - \gamma t\overline{(\Delta n)^2}$, so, even if the vacuum state is not present in the initial field state ($p_0 = 0$) and no photons are counted, the predicted mean photon number may be substantially less than $\bar{n}$ if $\overline{(\Delta n)^2} \gg \overline{n}$. Contrary to this, the mean number of photons in the absence of counts predicted by the E-model is

$$\widetilde{n} = \frac{\bar{n}}{1 + p_0(\widetilde{\gamma}t - 1)}, \qquad (18)$$

so, whenever $p_0 = 0$, the mean photon number remains unchanged, $\widetilde{n} = \bar{n}$.

3.3. Continuous counting

An obvious generalization of the SD superoperator for k-counts is

$$N_t(k) = \int_0^t \mathrm{d}t_k\, S_{t-t_k}\int_0^{t_k}\mathrm{d}t_{k-1}\, J_A S_{t_k-t_{k-1}}\cdots \times \int_0^{t_2}\mathrm{d}t_1\, J_A S_{t_1}. \qquad (19)$$

For any positive definite ρ satisfying $\mathrm{Tr}\,\rho = 1$, one has $0 \leqslant \mathrm{Tr}[N_t(k)\rho_0] \leqslant 1$ and $P(k,t) = \mathrm{Tr}[N_t(k)\rho_0]$ is the probability of counting (and removing) exactly k photons in a time t. Moreover, (19) shows a semigroup associative property,

$$N_{t_1+t_2}(k) = \sum_{k_1+k_2=k} N_{t_2}(k_2) N_{t_1}(k_1), \tag{20}$$

initial identity, $\lim_{t\to 0} N_t(0)\rho_0 = \rho_0$, and ideality: the operation $N_t(0) = S_t$ transforms pure states into pure states, noting that $S_t\rho_0$ is a pure state if ρ_0 is a pure state. After k photons have been counted, the original field state ρ_0 changes to *the conditioned state*

$$\rho_t^{(k)} \equiv \frac{N_t(k)\rho_0}{\mathrm{Tr}(N_t(k)\rho_0)}. \tag{21}$$

It is worth noting that the superoperator (19) also appears in the solution of the master equation (16). Indeed, this equation can be written as an 'inhomogeneous' partial differential equation

$$\partial\rho/\partial t + L_0\rho = J_A\rho \tag{22}$$

with

$$L_0\rho = \mathrm{i}\omega_0\big[a^\dagger a, \rho\big] + \tfrac{1}{2}\gamma_A\big\{AA^\dagger, \rho\big\} \tag{23}$$

and $J_A\rho$ understood as a 'source' term. Since the integral version of equation (22) is

$$\rho(t) = \mathrm{e}^{-L_0 t}\rho_0 + \int_0^t \mathrm{d}t'\, \mathrm{e}^{-(t-t')L_0} J_A \mathrm{e}^{-L_0 t'}\rho(t'),$$

the solution is obtained by successive iterations,

$$\rho(t) = S_t\rho_0 + \sum_{k=1}^{\infty}\int_0^t \mathrm{d}t_k\, S_{t-t_k}\int_0^{t_k}\mathrm{d}t_{k-1}\, J_A S_{t_k-t_{k-1}}\cdots \times \int_0^{t_2}\mathrm{d}t_1\, J_A S_{t_1}\rho_0,$$

where $S_t\bullet \equiv \mathrm{e}^{-L_0 t}\bullet = \mathrm{e}^{Y_A\tau}\bullet\,\mathrm{e}^{Y_A^\dagger\tau}$ is the free field evolution under continuous monitoring. So, we can write the field state at any time as

$$\rho(t) = T_t\rho_0 = S_t\rho_0 + \sum_{k=1}^{\infty} N_t(k)\rho_0. \tag{24}$$

The sum over all possible counted photons satisfies the normalization condition $\mathrm{Tr}[T_t\rho_0] = 1$, where

$$T_t = \sum_{k=0}^{\infty} N_t(k) = S_t + \int_0^t T_{t-t'} J_A S_{t'}\,\mathrm{d}t' \tag{25}$$

is the full evolution operator.

An explicit form of the superoperator N_t for the E-model is as follows:

$$\widetilde{N}_t(k)\rho_0 = \mathcal{U}_t\int_0^t \mathrm{d}t'\,\mathrm{e}^{-\widetilde{\gamma}t'}\frac{(t')^{k-1}}{(k-1)!}\exp\left[-\frac{\widetilde{\gamma}}{2}(t-t')\Lambda\right] \times \big(\widetilde{J}^k\rho_0\big)\exp\left[-\frac{\widetilde{\gamma}}{2}(t-t')\Lambda\right], \tag{26}$$

where

$$\mathcal{U}_t = \mathrm{e}^{-\mathrm{i}H_0 t}\bullet\,\mathrm{e}^{\mathrm{i}H_0 t} \tag{27}$$

is the free evolution superoperator.

The probability of k counts in the E-model equals

$$\widetilde{P}(k,t) = p_k\big[1 - \mathrm{e}^{-\widetilde{\gamma}t}\Omega(k,\widetilde{\gamma}t)\big] + \frac{\mathrm{e}^{-\widetilde{\gamma}t}(\widetilde{\gamma}t)^k}{k!}\sum_{n=k}^{\infty} p_n. \tag{28}$$

Here we have set

$$\Omega(k,x) = \sum_{j=0}^{k}\frac{x^j}{j!} = \frac{\mathrm{e}^x}{k!}\Gamma(k+1,x), \tag{29}$$

where

$$\Gamma(\alpha,x) = \int_x^\infty t^{\alpha-1}\mathrm{e}^{-t}\,\mathrm{d}t \tag{30}$$

is the complementary incomplete gamma function.

It can be verified that $\sum_{k=0}^{\infty}\widetilde{P}(k,t) = 1$. Formula (28) should be compared with the SD result (which, in turn, coincides with the results of [2, 6])

$$P(k,t) = \sum_{n=k}^{\infty}\binom{n}{k}(1-\mathrm{e}^{-\gamma t})^k(\mathrm{e}^{-\gamma t})^{n-k}p_n. \tag{31}$$

The limiting value, $\lim_{t\to\infty}\widetilde{P}(k,t) = \langle k|\rho_0|k\rangle = p_k$, in both equations, (28) and (31), means that, asymptotically, the counting statistics coincides with the photon statistics. This happens because the cavity is supposed ideal, and no photons are lost to the surroundings; whenever a photon is taken out of the field, it is registered and counted, with certainty.

4. Coincidence probabilities, waiting times and correlation functions

4.1. Coincidence probability density

The *coincidence probability density* $h_t^{(m)}(t_1,\ldots,t_m)$ (CPD), being multiplied by the product of infinitesimal time intervals $\Delta t_1\Delta t_2\cdots\Delta t_m$, gives the probability that one photocount is recorded in each of the non-overlapping intervals ($t_1 < t_2 < \cdots < t_m < t$)

$$[t_1, t_1+\Delta t_1), [t_2, t_2+\Delta t_2), \ldots, [t_m, t_m+\Delta t_m),$$

chosen by the experimentalist, *independently of how many photons were counted in between*. An obvious generalization of the expression for the CPD given in [4] reads (we suppress arguments of various functions when they are obvious)

$$h_t^{(m)} = \mathrm{Tr}[T_{t-t_m}J_A\cdots J_A T_{t_2-t_1}J_A T_{t_1}\rho_0], \tag{32}$$

recalling that T_{t_1} is the operator that evolves the field and the m J_A-superoperators stand for the counted photons at times $t_1, t_2, \ldots, t_m$.

In the SD model

$$T_t\rho_0 = \sum_{k=0}^{\infty} N_t(k)\rho_0 = S_t\sum_{n=0}^{\infty}\frac{\big(1-\mathrm{e}^{-\gamma t}\big)^n}{\gamma^n n!}J^n\rho_0, \tag{33}$$

with

$$JT_t\rho_0 = \mathrm{e}^{-\gamma t}T_t J\rho_0, \tag{34}$$

so the CPD depends on the sum $t_{1\leftrightarrow m} \equiv \sum_{k=1}^{m} t_k$ only (and does not depend on the time interval t):

$$\begin{aligned} h_t^{(m)} &= \mathrm{e}^{-\gamma t_{1\leftrightarrow m}} \mathrm{Tr}(T_t J^m \rho_0) \\ &= \gamma^m \mathrm{e}^{-\gamma t_{1\leftrightarrow m}} \mathrm{Tr}\left[\rho_0 \hat{n}(\hat{n}-1)\cdots(\hat{n}-m+1)\right] \\ &= m!\gamma^m \mathrm{e}^{-\gamma t_{1\leftrightarrow m}} \sum_{l=0}^{\infty} \binom{m+l}{m} p_{m+l}. \end{aligned} \quad (35)$$

Note that the series in (35) converges only for those sets of numbers $\{p_n\}$ which decrease faster than $(n-m)!/n! \sim n^{-m}$ for $n \to \infty$. Therefore, for many quantum states the CPDs simply do not exist in the SD model, especially for $m \gg 1$. Even when they do exist, their behaviour as functions of number of counts m turns out to be very sensitive to small changes of m, as shown in concrete examples below (see, e.g., figure 2). This drawback of SD theory is a direct consequence of the choice of the jump operator in the form (1): each time that operator J acts on the field density operator in equation (32), it not only shifts the photon number distribution $\{p_n\}$ by one quantum to the left, but it significantly increases (by the factor $\sqrt{n}$) the weight of the 'tail' of this distribution (members with $n \gg 1$). This change of the photon number distribution can be not very significant for one or two clicks ($m = 1, 2$), but for $m \gg 1$ the initial distribution becomes totally deformed, to such an extent that many probability densities cease to exist.

This nonphysical behaviour (i.e., an extreme sensitivity to the form of 'tail' of the distribution) is eliminated in nonlinear models in a quite natural manner, provided that the function $F(\hat{n})$ in equation (2) is chosen in such a way that it leads to some 'saturation' of the coefficient $F(n-1)\sqrt{n}$ (which now replaces the usual factor $\sqrt{n}$) for $n \gg 1$. In the most distinct form it is seen for the E-model with $F(n-1)\sqrt{n} = 1$. In this case, after some algebra, we get

$$\widetilde{J}\widetilde{T}_t \rho_0 = \mathrm{e}^{-\widetilde{\gamma} t} \mathcal{U}_t \mathrm{e}^{t\widetilde{J}} \widetilde{J} \rho_0, \quad (36)$$

where the operator $\mathcal{U}_t$ is given by (27). The sequential and alternate applications of $\widetilde{T}$ and $\widetilde{J}$ on ρ_0 lead to

$$\widetilde{J}\widetilde{T}_{t_m - t_{m-1}} \cdots \widetilde{J}\widetilde{T}_1 \rho_0 = \mathrm{e}^{-\widetilde{\gamma} t_m} \mathcal{U}_{t_m} \mathrm{e}^{t_m \widetilde{J}} \widetilde{J}^m \rho_0. \quad (37)$$

The RHS of (37) can also be written as $\widetilde{J}^m T_{t_m} \rho_0$, so the evolution depends on the time of the last count. Applying $\widetilde{T}_{t-t_m}$ on equation (37) we have

$$\begin{aligned} &\widetilde{T}_{t-t_m} \widetilde{J}\widetilde{T}_{t_m - t_{m-1}} \cdots \widetilde{J}\widetilde{T}_1 \rho_0 = \mathrm{e}^{-\widetilde{\gamma} t_m} T_{t-t_m} \mathcal{U}_{t_m} \mathrm{e}^{t_m \widetilde{J}} \widetilde{J}^m \rho_0 \\ &\quad = \mathrm{e}^{-\widetilde{\gamma} t_m} \mathcal{U}_t \Bigg\{ \mathrm{e}^{\widetilde{\gamma}(t-t_m)\Lambda/2} \mathrm{e}^{t_m \widetilde{J}} \widetilde{J}^m \rho_0 \mathrm{e}^{\widetilde{\gamma}(t-t_m)\Lambda/2} \\ &\qquad + \sum_{k=1}^{\infty} \int_0^{t-t_m} \mathrm{e}^{-\widetilde{\gamma} t'} \mathrm{e}^{-\widetilde{\gamma}(t-t_m-t')\Lambda/2} \mathrm{e}^{t_m \widetilde{J}} \widetilde{J}^{k+m} \rho_0 \\ &\qquad \times \mathrm{e}^{-\widetilde{\gamma}(t-t_m-t')\Lambda/2} \frac{t'^{k-1}}{(k-1)!} \mathrm{d}t' \Bigg\}. \end{aligned} \quad (38)$$

The calculation of the trace of equation (38) gives the CPD

$$\begin{aligned} \widetilde{h}_t^{(m)} &= \widetilde{\gamma}^m \Bigg\{ \mathrm{e}^{-\widetilde{\gamma} t_m} \sum_{l=0}^{\infty} \frac{(\widetilde{\gamma} t_m)^l}{l!} \sum_{k=0}^{\infty} p_{m+k+l} \\ &\quad + \Bigg[\sum_{k=0}^{\infty} \Omega(k, \widetilde{\gamma} t_m) \sum_{l=1}^{\infty} \frac{(\widetilde{\gamma}(t-t_m))^l}{l!} p_{m+k+l+1} \\ &\quad - \sum_{k=0}^{\infty} \Omega(k, \widetilde{\gamma}(t-t_m)) \sum_{l=0}^{\infty} \frac{(\widetilde{\gamma} t_m)^l}{l!} p_{m+k+l+1} \\ &\quad + \sum_{l=0}^{\infty} \Omega(l, \widetilde{\gamma} t_m) p_{m+l+1} \Bigg] \mathrm{e}^{-\widetilde{\gamma} t} \Bigg\}, \end{aligned} \quad (39)$$

where all series converge for any normalizable distribution $\{p_n\}$. Formula (39) can be simplified by setting $t_m = t$, meaning that the mth photocount, or click, 'turns off' the detector:

$$\widetilde{h}_t^{(m)}(t_1, \ldots, t_{m-1}, t) = \widetilde{\gamma}^m \mathrm{e}^{-\widetilde{\gamma} t} \sum_{l=0}^{\infty} \Omega(l, \widetilde{\gamma} t) p_{m+l}. \quad (40)$$

This CPD depends on the time of the last count only. Moreover, due to the property $\Omega(k, x) < \exp(x)$ for $x > 0$ and any k, which follows immediately from the definition (29), all CPDs in E-model are uniformly limited,

$$\widetilde{h}_t^{(m)}(t_1, \ldots, t_{m-1}, t) < \widetilde{\gamma}^m, \quad (41)$$

for arbitrary normalized field states (with $\sum_{k=0}^{\infty} p_k = 1$).

Below we give CPDs for a few field states.

(a) For the *coherent state* the probability of finding n photons in the field is $p_n = \exp[-\bar{n}]\bar{n}^n/n!$, so

$$h_t^{(m)} = \gamma^m \bar{n}^m \exp(-\gamma t_{1\leftrightarrow m}), \quad (42)$$

for the SD model, while in the E-model it is

$$\widetilde{h}_t^{(m)} = \widetilde{\gamma}^m \left[1 - \mathrm{e}^{-\widetilde{\gamma} t} \sum_{k=0}^{\infty} \frac{(\widetilde{\gamma} t)^l \, \Gamma(l+m, \bar{n})}{l!(l+m-1)!} \right]. \quad (43)$$

(b) For the *number or Fock state*, $p_n = \delta_{n,N}$, the SD model gives

$$h_t^{(m)} = \gamma^m \frac{N!}{(N-m)!} \exp(-\gamma t_{1\leftrightarrow m}), \quad (44)$$

and in the E-model,

$$\widetilde{h}_t^{(m)} = \widetilde{\gamma}^m \Omega(N-m, \widetilde{\gamma} t) \, \mathrm{e}^{-\widetilde{\gamma} t}. \quad (45)$$

(c) For the *thermal state* (13), the SD model gives

$$h_t^{(m)} = \gamma^m m! \bar{n}^m \exp(-\gamma t_{1\leftrightarrow m}), \quad (46)$$

while in the E-model, formulae (39) and (40) lead to the same result,

$$\widetilde{h}_t^{(m)} = \widetilde{\gamma}^m \left(\frac{\bar{n}}{\bar{n}+1} \right)^m \mathrm{e}^{-\widetilde{\gamma} t/(\bar{n}+1)}. \quad (47)$$

The coincidence probabilities of the SD model, following from equations (42), (44) and (46), may assume quite large values even for small values of $\gamma \Delta t$, while those following from equations (43), (45) and (47) are always less than 1 if $\widetilde{\gamma} \Delta t < 1$, independently of the value of $\bar{n}$.

In figure 1 we have plotted the CPD divided by $\widetilde{\gamma}^m$, as a function of $\widetilde{\gamma} t$, for the E-model. We note that no curve exceeds the value 1. We did not plot the analogous curves for the SD model, because in this case they all have exponential decay as a function of the sum of all detection times.

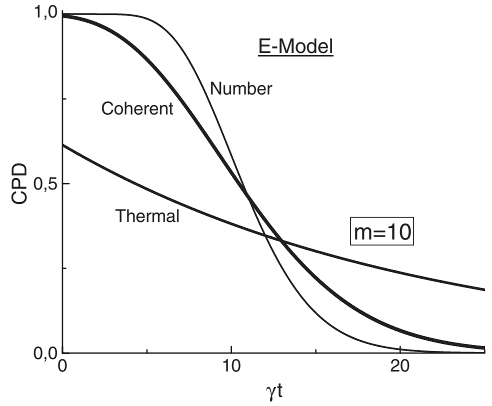

Figure 1. The normalized (divided by $\widetilde{\gamma}^m$) CPD $\widetilde{h}_t^{(m)}(t_1, t_2, \ldots, t_m)$ for the E-model, see equation (40), as a function of $\widetilde{\gamma}t$ (with $t = t_m$), for three field states: number state, coherent state, and thermal state. The parameter $m = 10$ and the mean photon number $\bar{n} = 20$ are the same for the three states. The CPD is always less than 1 and it decreases monotonically.

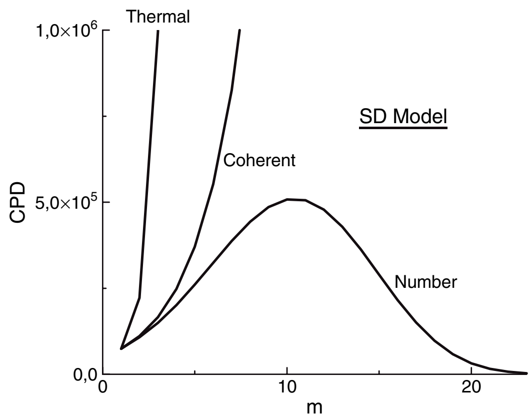

Figure 2. The normalized (divided by γ^m) CPD $h_t^{(m)}(t_1, t_2, \ldots, t_m)$ for the SD model, see equation (35), as a function of m, for three field states: number state, coherent state, and thermal state. We have considered equal spacings, $t_k = k\tau$, $k = 1, 2, \ldots, m$. The parameter $\gamma t_m = 6$ and the mean photon number $\bar{n} = 30$ are the same for the three states. Note that the CPD can assume very large values.

In figure 2 we have plotted the CPD divided by γ^m, now as a function of the number of counts m, for the SD model, assuming that $t_k = kt_1$, $k = 1, 2, \ldots, m$ and that the time duration $t_m = mt_1$ is fixed (obviously, m can assume only integral values; however, we joined the points for convenience). We see that the CPD can attain huge values in the SD theory, especially for thermal states, due to the presence of the term $\bar{n}^m m!$ in equation (46), even for not very high values of m (for example, for $m = 10$ and $\bar{n} = 30$ the factor $m!\bar{n}^m$ is of the order of 10^{21}). Moreover, the behaviour of the CPD in SD model is very sensitive to a 'competition' between the factorial and exponential terms. Indeed, for an equal time spacing, $t_k = kt_1$, equation (46) assumes the form

$$h_t^{(m)}(t_1, 2t_1, \ldots, mt_1)_{\mathrm{therm}} = m!\,\mathrm{e}^{-\gamma t_m(m+1)/2}(\gamma\bar{n})^m,$$

whereas in the coherent state we obtain the same expression, but without the factorial:

$$h_t^{(m)}(t_1, 2t_1, \ldots, mt_1)_{\mathrm{coh}} = \mathrm{e}^{-\gamma t_m(m+1)/2}(\gamma\bar{n})^m.$$

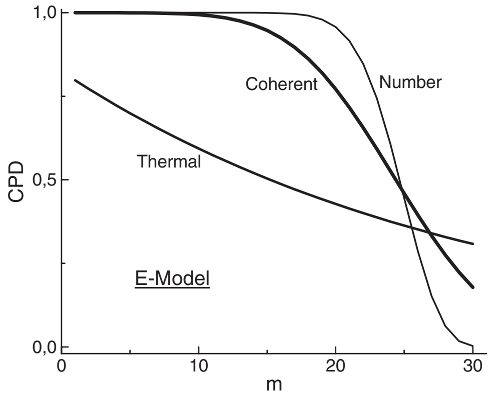

Figure 3. The normalized (divided by $\widetilde{\gamma}^m$) CPD $\widetilde{h}_t^{(m)}(t_1, t_2, \ldots, t_m)$ for the E-model, see equation (40), as a function of m, for three field states: number state, coherent state, and thermal state. The parameter $\widetilde{\gamma}t_m = 7$ and the mean photon number $\bar{n} = 30$ are the same for the three states. The CPD is always less than 1 and it decreases monotonically.

Thus even small variations of the parameters (e.g., γt_m in the above examples) can result in drastic changes of the qualitative behaviour of the CPD. For example, in the case of the coherent state, an exponential increase of the CPD for $\gamma\bar{n}\mathrm{e}^{-\gamma t_m/2} > 1$ goes to an exponential decrease, if $\gamma\bar{n}\mathrm{e}^{-\gamma t_m/2} < 1$, while the CPD of the thermal state continues to increase due to the presence of the factorial term. In contrast to this, all normalized CPDs of the E-model do not exceed the unit value, as seen in figure 3.

4.2. The two-count conditional probability density

The *two-count conditional probability density* (TCCP),

$$C(t_1 + \tau|t_1) := \frac{\mathrm{Tr}(J T_\tau J T_{t_1}\rho_0)}{\mathrm{Tr}(J T_{t_1}\rho_0)}, \tag{48}$$

means that if, with certainty, one photon has been counted at time t_1, $C(t_1 + \tau|t_1)\Delta\tau$ is the probability that another photon is counted in the interval between $t_1 + \tau$ and $t_1 + \tau + \Delta\tau$, independently of the number of counts in between. In order to simplify formulae, we shall use the short notation $t_{1+\tau}$ instead of $t_1 + \tau$. In the SD model, the TCCP depends only on the quantity $t_{1+\tau}$ and does not depend on t_1:

$$C(t_{1+\tau}|t_1) = \frac{\gamma\mathrm{e}^{-\gamma t_{1+\tau}}\sum_{n=0}^{\infty}(n+1)(n+2)\,p_{n+2}}{\sum_{n=0}^{\infty}(n+1)\,p_{n+1}},$$

while in the E-model the TCCP depends on all three parameters, t_1, τ and $t_{1+\tau}$:

$$\widetilde{C}(t_{1+\tau}|t_1) = \frac{\widetilde{\gamma}\mathrm{e}^{-\widetilde{\gamma}\tau}\sum_{n=0}^{\infty}\Omega(n, \widetilde{\gamma}t_{1+\tau})p_{n+2}}{\sum_{n=0}^{\infty}\Omega(n, \widetilde{\gamma}t_1)p_{n+1}}. \tag{49}$$

For three different field states we have the following explicit formulae (we suppress the argument $(t_{1+\tau}|t_1)$ of the TCCP).

(a) *Coherent states*:

$$C = \gamma\bar{n}\mathrm{e}^{-\gamma t_{1+\tau}}, \tag{50}$$

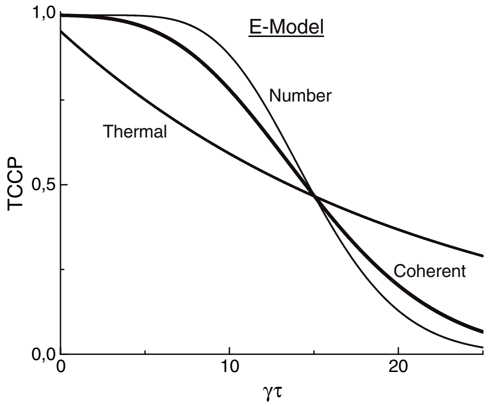

Figure 4. The normalized (divided by $\widetilde{\gamma}$) TCCP as a function of $\widetilde{\gamma}\tau$ for the E-model, for three field states: number state, coherent state, and thermal state. We have set $\widetilde{\gamma}t_1 = 4$ and $\bar{n} = N = 20$ in equations (51), (53) and (55).

$$\widetilde{C} = \widetilde{\gamma}\,\frac{1 - \mathrm{e}^{-\widetilde{\gamma}t_{1+\tau}}\sum_{l=0}^{\infty}\dfrac{(\widetilde{\gamma}t_{1+\tau})^l}{l!(l+1)!}\Gamma(l+2,\bar{n})}{1 - \mathrm{e}^{-\widetilde{\gamma}t_1}\sum_{l=0}^{\infty}\dfrac{(\widetilde{\gamma}t_1)^l}{l!^2}\Gamma(l+1,\bar{n})}. \qquad (51)$$

(b) *Number states* $|N\rangle$:

$$C = \gamma(N-1)\mathrm{e}^{-\gamma t_{1+\tau}}, \qquad (52)$$

$$\widetilde{C} = \widetilde{\gamma}\,(N-1)\,\frac{\Gamma(N-1,\widetilde{\gamma}t_{1+\tau})}{\Gamma(N,\widetilde{\gamma}t_1)}. \qquad (53)$$

(c) *Thermal states*:

$$C = 2\gamma\bar{n}\mathrm{e}^{-\gamma t_{1+\tau}}, \qquad (54)$$

$$\widetilde{C} = \widetilde{\gamma}\frac{\bar{n}}{\bar{n}+1}\mathrm{e}^{-\widetilde{\gamma}\tau/(\bar{n}+1)}. \qquad (55)$$

In figure 4 are plotted the TCCPs divided by $\widetilde{\gamma}$ as a function of $\widetilde{\gamma}\tau$ for the E-model, that show different behaviour for each field state. We did not plot the curves for the SD model, because they show qualitatively the same exponential decay. Note, however, that in the E-model TCCP$\times\Delta\tau < 1$ always (at least for quantum states considered), while in the SD model this product can exceed 1 for small or moderate values of $\gamma\tau$, if the mean number of photons $\bar{n}$ is big enough (remember that $\gamma\Delta\tau$ must be less than 1).

4.3. Waiting times

We can also compare expressions for two *waiting times* [13, 35]. The *unconditional waiting time* τ_{u} has the following meaning: picking an arbitrary time t_1 during the counting process, τ_{u} is the average time for the first arriving photon, or first click, of the counter. Actually, it is a one-count probability function. The *conditional waiting time* τ_{c} is the average time between two consecutive clicks, one at t_1 and the next at t_2, *with no counts in between* (a two-count probability but different from the TCCP). We can calculate these characteristic times, using two unnormalized (the time as the random variable) probability densities: the *unconditional probability count density* (UPC)

$$W_{\mathrm{u}}(t_{1+\tau}|t_1) = \mathrm{Tr}[J S_\tau T_{t_1}\rho_0] \qquad (56)$$

and the *conditional probability count density* (CPC)

$$W_{\mathrm{c}}(t_{1+\tau}|t_1) = \frac{\mathrm{Tr}\left[J S_\tau J T_{t_1}\rho_0\right]}{\mathrm{Tr}\left[J T_{t_1}\rho_0\right]}. \qquad (57)$$

So, the unconditional and conditional waiting times are calculated as

$$\tau_{\mathrm{u,c}} = \frac{\int_0^\infty \tau W_{\mathrm{u,c}}(t_1, t_{1+\tau})\,\mathrm{d}\tau}{\int_0^\infty W_{\mathrm{u,c}}(t_1, t_{1+\tau})\,\mathrm{d}\tau}. \qquad (58)$$

In the SD model

$$W_{\mathrm{u}} = \gamma\mathrm{e}^{-\gamma t_{1+\tau}}\sum_{n=1}^{\infty} n p_n(1 - \mathrm{e}^{-\gamma t_1} + \mathrm{e}^{-\gamma t_{1+\tau}})^{n-1}, \qquad (59)$$

$$W_{\mathrm{c}} = \frac{\gamma}{\bar{n}}\mathrm{e}^{-\gamma t_{1+\tau}}\sum_{n=0}^{\infty}(n+1)(n+2) \times (1 - \mathrm{e}^{-\gamma t_1} + \mathrm{e}^{-\gamma t_{1+\tau}})^n p_{n+2}, \qquad (60)$$

while in the E-model

$$\widetilde{W}_{\mathrm{u}} = \widetilde{\gamma}\mathrm{e}^{-\widetilde{\gamma}t_{1+\tau}}\sum_{n=0}^{\infty}\Omega\,(n, \widetilde{\gamma}t_1)\,p_{n+1}, \qquad (61)$$

$$\widetilde{W}_{\mathrm{c}} = \widetilde{\gamma}\mathrm{e}^{-\widetilde{\gamma}\tau}\frac{\sum_{n=0}^{\infty}\Omega\,(n, \widetilde{\gamma}t_1)p_{n+2}}{\sum_{n=0}^{\infty}\Omega\,(n, \widetilde{\gamma}t_1)p_{n+1}}. \qquad (62)$$

Note that the two-count conditional probability densities, $\widetilde{C}(t_{1+\tau}|t_1)$ and $\widetilde{W}_{\mathrm{c}}(t_{1+\tau}|t_1)$, only differ in the time dependence of the Ω-function, present in their numerator. While in equation (49) it depends on $t_{1+\tau}$, in equation (62) the dependence is on t_1. Moreover, the dependence on τ in the probabilities (61) and (62) appears only as an exponent in a factor multiplying a function that depends solely on t_1. Some explicit expressions for unconditional and conditional probability densities for the two models are given in the appendix.

In the E-model, the waiting times are calculated by doing the substitutions $W_{\mathrm{u,c}} \to \widetilde{W}_{\mathrm{u,c}}$ in equation (58). One can verify that the conditional and unconditional times are the same for any field state:

$$\widetilde{\tau}_{\mathrm{u}} = \widetilde{\tau}_{\mathrm{c}} = \widetilde{\gamma}^{-1}, \qquad (63)$$

a property that characterizes a random stationary process.

For the SD model the waiting times depend on the field state; see [13].

(a) *Coherent state*:

$$\tau_{\mathrm{u}} = \tau_{\mathrm{c}} = \frac{\Gamma\left(0, \bar{n}\mathrm{e}^{-\gamma t_1}\right) - \ln(\bar{n}\mathrm{e}^{-\gamma t_1}) - \gamma'}{\gamma\left[\exp(\bar{n}\mathrm{e}^{-\gamma t_1}) - 1\right]}, \qquad (64)$$

where $\gamma' = 0.577\,21\ldots$ is the Euler constant. The waiting times coincide, because the respective probabilities are the same.

(b) *Number state* $|N\rangle$:

$$\tau_{\mathrm{u}} = \frac{\sum_{n=1}^{N} n^{-1}\binom{N}{n}\left(1 - \mathrm{e}^{-\gamma t_1}\right)^{N-n}\left(\mathrm{e}^{-\gamma t_1}\right)^n}{\gamma\left[1 - (1 - \mathrm{e}^{-\gamma t_1})^N\right]},$$

$$\tau_{\mathrm{c}} = \frac{\sum_{n=1}^{N-1} n^{-1}\binom{N-1}{n}\left(1 - \mathrm{e}^{-\gamma t_1}\right)^{N-n-1}\left(\mathrm{e}^{-\gamma t_1}\right)^n}{\gamma\left[1 - (1 - \mathrm{e}^{-\gamma t_1})^{N-1}\right]}.$$

For $t_1 = 0$, $\tau_{\mathrm{u}} = (\gamma N)^{-1}$ and $\tau_{\mathrm{c}} = [\gamma(N-1)]^{-1}$.

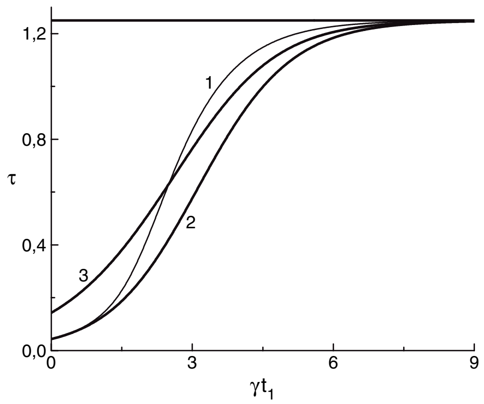

Figure 5. The waiting times as a function of γt_1 for the SD model and for the E-model, in the case of $\gamma = \tilde{\gamma} = 0.8$. The horizontal line corresponds to the E-model, whereas curves 1–3 are related to the SD model. Curve 1 shows coinciding conditional and unconditional times for the coherent state, curve 2 the conditional time for the thermal state, and curve 3 the unconditional time for the thermal state. In all the cases we have set the mean photon number $\bar{n} = 30$.

(c) *Thermal state*:

$$\tau_\mathrm{u} = \frac{\ln(1+\bar{n}\mathrm{e}^{-\gamma t_1})}{\gamma \bar{n}\mathrm{e}^{-\gamma t_1}}, \qquad \tau_\mathrm{c} = \frac{\bar{n}\mathrm{e}^{-\gamma t_1} + \ln(1+\bar{n}\mathrm{e}^{-\gamma t_1})}{\gamma \bar{n}\mathrm{e}^{-\gamma t_1}\,(\bar{n}\mathrm{e}^{-\gamma t_1}+2)}.$$

For $t_1 = 0$,

$$\tau_\mathrm{u} = \frac{\ln(1+\bar{n})}{\gamma \bar{n}}, \qquad \tau_\mathrm{c} = \frac{\bar{n} + \ln(1+\bar{n})}{\gamma \bar{n}\,(\bar{n}+2)}.$$

Thus, important differences between the two models occur in the estimated waiting times: while in the SD model they depend significantly on the field state, in the E-model they are insensitive to it. For comparison, in figure 5 we have plotted the waiting times as a function of γt_1, assuming that $\tilde{\gamma} = \gamma$. The horizontal line corresponds to a constant (independent of the quantum state) waiting time in the E-model. In the SD model, all waiting times tend to the same limiting value γ^{-1}, provided that the product $\bar{n}\mathrm{e}^{-\gamma t_1}$ is small enough (tends to zero). If $t_1 = 0$, then the predictions of the SD model and the E-model concerning waiting times approximately coincide for quantum states with $\bar{n} \ll 1$ (because in this case the effects of nonlinearity are small, so that the operators a and E_- approximately coincide for such states). Also, both models give close results in the opposite limiting case of highly excited states with $\bar{n} \gg 1$ (and $t_1 = 0$), if one 'renormalizes' the coefficient $\tilde{\gamma}$, taking $\tilde{\gamma} \approx \gamma\bar{n}$ (which also agrees with the actions of the operators a and E_- on such states).

4.4. The second order correlation function

The normalized second order temporal coherence function

$$g^{(2)}(\tau) = \frac{h(t_1, t_{1+\tau})}{[h(t_1)]^2} = \frac{\mathrm{Tr}\left[J T_\tau J T_{t_1}\rho_0\right]}{\left(\mathrm{Tr}\left[J T_{t_1}\rho_0\right]\right)^2} \tag{65}$$

tells us about photon bunching ($g^{(2)}(\tau) < g^{(2)}(0)$) or antibunching ($g^{(2)}(\tau) > g^{(2)}(0)$). In the SD model,

$$g^{(2)}(\tau) = g^{(2)}(0)\mathrm{e}^{-\gamma\tau}, \qquad g^{(2)}(0) = \frac{\overline{n^2} - \bar{n}}{\bar{n}^2}, \tag{66}$$

for any field state, so only the phenomenon of photon bunching is possible. In the E-model,

$$\tilde{g}^{(2)}(\tau) = \mathrm{e}^{\tilde{\gamma}(t_1-\tau)} \frac{\sum_{n=0}^{\infty} \Omega\left(n, \tilde{\gamma}(t_1+\tau)\right) p_{n+2}}{\left[\sum_{n=0}^{\infty} \Omega\left(n, \tilde{\gamma} t_1\right) p_{n+1}\right]^2} \tag{67}$$

depends on the field state, as can be seen in the three examples below.

(a) *Coherent state*:

$$\tilde{g}^{(2)}(\tau) = \mathrm{e}^{\tilde{\gamma}(t_1-\tau)+\bar{n}} \frac{\sum_{n=0}^{\infty} (\bar{n}^n/(n+2)!)\,\Omega\left(n, \tilde{\gamma}(t_1+\tau)\right)}{\left[\sum_{n=0}^{\infty} (\bar{n}^n/(n+1)!)\,\Omega\left(n, \tilde{\gamma} t_1\right)\right]^2}.$$

(b) *Number state*:

$$\tilde{g}^{(2)}(\tau) = \mathrm{e}^{\tilde{\gamma}(t_1-\tau)} \frac{\Omega\left(N-2, \tilde{\gamma}(t_1+\tau)\right)}{\left[\Omega\left(N-1, \tilde{\gamma} t_1\right)\right]^2}.$$

(c) *Thermal state*:

$$\tilde{g}^{(2)}(\tau) = \tilde{g}^{(2)}(0)\mathrm{e}^{-\tilde{\gamma}\tau/(\bar{n}+1)}, \qquad \tilde{g}^{(2)}(0) = \mathrm{e}^{\tilde{\gamma} t_1/(\bar{n}+1)}.$$

Calculations show that $\tilde{g}^{(2)}(\tau) < \tilde{g}^{(2)}(0)$ for these three field states, so the phenomenon of photon bunching also prevails in the E-model.

Interestingly, by comparing the behaviour of waiting times and the normalized second order correlation functions, we note that while for the SD model the waiting times depend on the field state and in the E-model they do not, the opposite occurs for the normalized second order correlation functions: in the SD model they do not depend on the field state while they do depend in the E-model.

5. Summary, discussion and conclusions

Srinivas and Davies have identified an inconsistency, quoted in their paper, that also occurs in other continuous photocount models, when the quantum jump superoperator as defined in equation (1) is used in the theory. Here we have proposed a modification of the SD model, consisting in replacing the usual 'lowering' and 'rising' operators a and $a^\dagger$ by nonlinear operators (2). We have demonstrated how naturally nonlinear operators can be introduced into the theory of photodetection, analysing the time evolution of a microscopic model for the interaction of a set of two-level atoms, that mimic a detector, with a one-mode field. We concentrated our analysis on the concrete example of the 'E-model', defining the new quantum jump superoperator with the aid of the nonlinear 'exponential phase operators' E_- and E_+ (3). We have shown explicitly that no inconsistencies arise in this model; in particular, all multitime probability densities exist and are well behaved for arbitrary quantum states, in contrast with the SD theory.

From the point of view of physics, the E-model can be considered, in a certain sense, as a limiting case of a large family of possible admissible nonlinear modifications of the SD scheme, which corresponds, presumably, to high intensities of the field and a saturation of the counting rate. For this reason, some specific predictions of the E-model (such as, for example, a nonexponential decrease of number of photons in the cavity due to their continuous counting [27], or independence of waiting times on the field state) are very different from the predictions of the SD model, which works well (as was shown in numerous applications of this model) for relatively low intensities. Obviously, the SD model is a weak nonlinearity limit of nonlinear models. Therefore, it would be very interesting to analyse the whole family of nonlinear jump operators with different functions $F(\hat{n})$ in equation (2), in order to find some 'continuous transition' from the SD model to the E-model. We hope to report on this problem elsewhere.

Many preceding studies of different phenomena, such as quantum non-demolition measurements [8], determination of field states under continuous photodetection process [10, 12], quantum theory of field-quadrature measurements [36], generation of special states of field via continuous measurements [37] or control of the amount of entanglement between two fields [38], have made use of the SD theory of photodetection. We believe that applications within its different nonlinear modifications (including the present E-model) may, indeed, bring new insight for both the quantum measurement theory and experiment.

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Appendix. Conditional probability functions

Here we present explicit expressions for the conditional and unconditional probability densities (59)–(62) for the three field states.

(a) *Coherent state*:

$$W_{\rm u} = W_{\rm c} = \gamma\bar{n}{\rm e}^{-\gamma(t_1+\tau)}\exp\left[-\bar{n}{\rm e}^{-\gamma t_1}\left(1-{\rm e}^{-\gamma\tau}\right)\right],$$

$$\widetilde{W}_{\rm u} = \widetilde{\gamma}{\rm e}^{-\widetilde{\gamma}\tau}\left[1-{\rm e}^{-\widetilde{\gamma}t_1}\sum_{l=0}^{\infty}\frac{(\widetilde{\gamma}t_1)^l}{l!^2}\Gamma(l+1,\bar{n})\right],$$

$$\widetilde{W}_{\rm c} = \widetilde{\gamma}{\rm e}^{-\widetilde{\gamma}\tau}\,\frac{1-{\rm e}^{-\widetilde{\gamma}t_1}\sum_{l=0}^{\infty}\frac{(\widetilde{\gamma}t_1)^l}{l!(l+1)!}\Gamma(l+2,\bar{n})}{1-{\rm e}^{-\widetilde{\gamma}t_1}\sum_{l=0}^{\infty}\frac{(\widetilde{\gamma}t_1)^l}{l!^2}\Gamma(l+1,\bar{n})}.$$

(b) *Number state*:

$$W_{\rm u} = \gamma N{\rm e}^{-\gamma t_{1+\tau}}\left[1-{\rm e}^{-\gamma t_1}\left(1-{\rm e}^{-\gamma\tau}\right)\right]^{N-1},$$

$$W_{\rm c} = \gamma{\rm e}^{-\gamma t_{1+\tau}}(N-1)\left[1-{\rm e}^{-\gamma t_1}\left(1-{\rm e}^{-\gamma\tau}\right)\right]^{N-2},$$

$$\widetilde{W}_{\rm u} = \widetilde{\gamma}\Omega(N-1,\widetilde{\gamma}t_1){\rm e}^{-\widetilde{\gamma}t_{1+\tau}},$$

$$\widetilde{W}_{\rm c} = \widetilde{\gamma}\frac{\Omega(N-2,\widetilde{\gamma}t_1)}{\Omega(N-1,\widetilde{\gamma}t_1)}{\rm e}^{-\widetilde{\gamma}\tau}.$$

(c) *Thermal state*:

$$W_{\rm u} = \frac{\gamma\bar{n}{\rm e}^{-\gamma t_{1+\tau}}}{[1+\bar{n}{\rm e}^{-\gamma t_1}(1-{\rm e}^{-\gamma\tau})]^2},$$

$$W_{\rm c} = \frac{2\gamma\bar{n}{\rm e}^{-\gamma t_{1+\tau}}}{[1+\bar{n}{\rm e}^{-\gamma t_1}(1-{\rm e}^{-\gamma\tau})]^3},$$

$$\widetilde{W}_{\rm u} = \widetilde{\gamma}\frac{\bar{n}}{\bar{n}+1}{\rm e}^{-\widetilde{\gamma}t_1/(\bar{n}+1)}{\rm e}^{-\widetilde{\gamma}\tau},$$

$$\widetilde{W}_{\rm c} = \widetilde{\gamma}\frac{\bar{n}}{\bar{n}+1}{\rm e}^{-\widetilde{\gamma}\tau}.$$

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