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Dynamics of entanglement between field modes in a one-dimensional cavity with a vibrating boundary

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Abstract
We study the dynamics of the *coefficient of quantum separability* between resonantly coupled modes of the massless scalar field in a one-dimensional ideal cavity, whose boundary performs small harmonic oscillations at the frequency $\omega_w = p\omega_1$ (where ω_1 is the fundamental field eigenfrequency). A qualitative difference between the cases $p = 1$ and $p = 2$ is discovered. In the first case, initially factorized thermal states remain separable in the process of evolution, whereas the dynamics of entanglement of initial entangled states (modelled by *two-mode squeezed states*) turns out to be more intricate, showing sudden changes from 'classical' to 'nonclassical' regimes. In contrast, in the 'principal resonance case' $p = 2$ the degree of entanglement changes with time monotonically, and any initially factorized thermal states go asymptotically to mixed quantum states with maximal degree of entanglement. Moreover, for highly thermalized states, the time dependence of the separability coefficient exhibits sharp jumps, which are, nonetheless, not correlated with time dependences of the energies of the field modes.

Keywords: nonstationary Casimir effect, vibrating boundary, parametric resonance, coupled modes, entanglement, separability, covariances, Gaussian states, thermal states, two-mode squeezed states

1. Introduction

Studies of quantum and classical fields in regions confined with oscillating boundaries are interesting for many reasons. One of them is the possibility of the fascinating effect of photon generation from the vacuum due to the nonstationary (dynamical) Casimir effect [1–3] (for other references, see the review [4]). Another reason is the existence of interesting mathematical structures and solutions, discovered in connection with this problem; see, e.g., [5–7] and references therein. In particular, under certain conditions, the dynamics of field modes can be described by an effective *quadratic nonstationary Hamiltonian* of the form [8–10]

$$H = \frac{1}{2}\sum_{\alpha}[p_\alpha^2 + \omega_\alpha^2(t)q_\alpha^2] + \sum_{\alpha\neq\beta} p_\alpha\gamma_{\alpha\beta}(t)q_\beta. \qquad (1)$$

Its interesting feature is the unusual 'momentum–coordinate' coupling, whose strength (antisymmetric coefficients $\gamma_{\alpha\beta}(t)$) is proportional to the velocity of the moving boundary (for a special case, coupling of such a form was considered earlier in [11]).

A general theory of nonstationary quantum systems with quadratic Hamiltonians (and a *finite* number of degrees of freedom) was developed in [12, 13]. In the context of the nonstationary Casimir effect, a few special cases of the

Hamiltonian (1) were studied previously. A single resonant mode was considered in [1, 2] for harmonic oscillations of a boundary and in [14] for an arbitrary *periodic* motion. Different examples of two resonant modes were analysed in [10, 15, 16]. However, most of the theoretical studies were connected with the model of a one-dimensional (1D) Fabry–Perot cavity, where an infinite number of modes can be excited due to the *equidistant* nature of the spectrum of eigenmodes in a stationary cavity; see [5, 17, 18] for analytical solutions and [4] for many other references.

It appears that the dynamics of quantum fields in one-dimensional cavities with oscillating boundaries is rather different from that in the case of three (or two) dimensions (where the eigenmode spectra are not equidistant). One of the differences is that although the total energy of the field grows exponentially under resonance conditions, both in the 1D and 3D cavities the number of photons in each mode of the 1D cavity increases with time only linearly (and the total number of photons grows quadratically). It was pointed out in [2] that this difference is due to strong intermode interaction in the 1D case (whose physical reason is the Doppler effect). Also, just due to interference between modes with an equidistant spectrum, the energy density of the field is concentrated in the form of sharp packets in the one-dimensional case [18–22].

The aim of this paper is to give a *quantitative* description of the degree of correlations between different modes in the 1D cavity. That is, taking into account a burst of interest in the phenomenon of *entanglement*, observed in the past few years (see general reviews [23] and [24–26] in the context of moving mirror problems), we address the following questions: (i) how we can numerically characterize the degree of correlation between the modes, (ii) how we distinguish 'quantum' correlations from 'classical' ones, and (iii) how the degree of 'entanglement' evolves with time for different kinds of parametric resonances between the field modes and the motion of cavity boundary. It was shown in previous papers that the cases where the frequency of the boundary coincides with the fundamental field mode eigenfrequency (named 'semiresonance' in [5]) and where it equals the doubled fundamental frequency ('principal resonance') are essentially different. In the semiresonance case, the total number of quanta in all modes is preserved in time (so photon creation from the vacuum is impossible [27, 28]), and initial excitations of the low frequency modes are transferred to high frequency ones. Thus, the initial state of the field is important for the dynamics. In contrast, in the principal resonance case, the initial conditions influence the dynamics only at the beginning of the process, whereas the asymptotical behaviour of the energy and number of photons in the field modes is determined mainly by the excitation of the vacuum state [5, 17]. Here we show that qualitative differences between the two cases are also observed when one studies the dynamics of entanglement.

The plan of the paper is as follows. In section 2 we introduce a *separability coefficient*, which tells us whether the correlations between the modes are 'classical' or 'quantum'. In section 3 we bring in some results from our previous papers [5, 17, 18], which are necessary for calculating this coefficient for the field modes in a 1D cavity with oscillating boundaries. The elements of covariance matrices connecting different field modes (which the separability coefficient depends on) are given in section 4, where the principal resonance case is analysed. Section 5 is devoted to the analysis of the semiresonance case. Section 6 contains conclusions. Some details of calculations and explicit expressions for various intermediate functions are collected in three appendices.

2. The separability coefficient

Let us consider a quantum system which can be described in terms of standard bosonic lowering/raising operators $\hat{a}_k$, $\hat{a}_k^\dagger$ or equivalent 'quadrature component' operators (we assume $\hbar = 1$):

$$\hat{a}_k = (\omega_k \hat{x}_k + \mathrm{i}\hat{p}_k)/\sqrt{2\omega_k} \qquad k = 1, 2, \ldots \tag{2}$$

where ω_k is the frequency of the kth mode. Assuming that all mean values of operators $\hat{a}_k$ equal zero (otherwise it is sufficient to replace operators $\hat{a}_k$ by $\hat{a}_k - \langle \hat{a}_k \rangle$), we define symmetrical real covariances between the kth and jth modes as follows:

$$q_{\alpha\beta} = q_{\beta\alpha} \equiv \tfrac{1}{2}\langle \hat{q}_\alpha \hat{q}_\beta + \hat{q}_\beta \hat{q}_\alpha \rangle \equiv \widetilde{q_\alpha q_\beta} \tag{3}$$

where the q_α are the components of the four-dimensional vector $\mathbf{q} = (x_k, p_k, x_j, p_j)$. It is convenient to gather the covariances in the symmetrical 4×4 covariance matrix $\mathcal{Q}$, dividing this matrix into 2×2 blocks as follows:

$$\mathcal{Q} = \|q_{\alpha\beta}\| = \left\| \begin{matrix} \mathcal{Q}_{kk} & \mathcal{Q}_{kj} \\ \mathcal{Q}_{jk} & \mathcal{Q}_{jj} \end{matrix} \right\| \qquad \mathcal{Q}_{kj} = \tilde{\mathcal{Q}}_{jk} \tag{4}$$

(a tilde over a matrix means matrix transposition).

One of the intriguing questions in the theory of quantum entangled systems is how to distinguish *quantum* correlations between different modes from 'classical' ones [23]. It is formulated usually as the problem of *separability* [29] of mixed quantum states, i.e., the possibility of representing the statistical operator $\hat{\rho}$ of the total system as a sum of direct products of statistical operators acting on each mode (say, 1 and 2) separately:

$$\hat{\rho} = \sum_i p_i \hat{\rho}_{i1} \otimes \hat{\rho}_{i2} \qquad p_i \geqslant 0 \quad \sum_i p_i = 1. \tag{5}$$

This problem was solved for bipartite continuous variable *Gaussian* states [23]. For our purposes, the most convenient criterion of separability is that found in [30], because it is expressed directly in terms of invariants of blocks of the covariance matrix (4). According to [30], the necessary and sufficient condition for separability of a Gaussian state of coupled kth and jth modes is the fulfilment of the inequality

$$I_k I_j + (|I_{kj}| - 1/4)^2 - I_4 - (I_k + I_j)/4 \geqslant 0 \tag{6}$$

where

$$I_k = \det \mathcal{Q}_{kk} \qquad I_{kj} = \det \mathcal{Q}_{kj}$$

$$I_4 = \mathrm{Tr}(\mathcal{Q}_{kk}\Sigma\mathcal{Q}_{kj}\Sigma\mathcal{Q}_{jj}\Sigma\mathcal{Q}_{jk}\Sigma) \qquad \Sigma = \left\| \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \right\|.$$

Taking into account that the term I_4, given by the trace of the product of eight matrices, is in fact incorporated in the determinant of the total covariance matrix, due to the identity [30] $\det \mathcal{Q} = I_k I_j + I_{kj}^2 - I_4$, the separability criterion (6) can be written in a simpler form [31]: $\mathcal{A}_{kj} \geqslant 0$, where (for $k \neq j$)

$$\mathcal{A}_{kj} \equiv \det \mathcal{Q} - \tfrac{1}{2}|\det \mathcal{Q}_{kj}| + \tfrac{1}{16} - \tfrac{1}{4}(\det \mathcal{Q}_{kk} + \det \mathcal{Q}_{jj}). \quad (7)$$

The quantity $\mathcal{A}_{kj}$ is positive for separable ('classical') states and negative for nonseparable ('genuine entangled') quantum states. However, it is not limited either from below or from above. Therefore we define the *parameter of separability* between the kth and jth modes as

$$\mathcal{S}_{kj} = \tanh(\mathcal{A}_{kj}). \quad (8)$$

Then negative values of $\mathcal{S}_{kj}$ in the interval $(-1, 0)$ correspond to entangled ('nonclassical') states of the two-mode system (we see some analogy with Mandel's Q-parameter for one-mode systems, where the values of Q belonging to the same interval $(-1, 0)$ are usually associated with 'nonclassicality' of a state; the difference is that the value -1 for $\mathcal{S}_{kj}$ cannot be achieved).

Of course, one can choose instead of (8) many other monotonic functions $F(\alpha\mathcal{A}_{kj})$, whose values belong to the interval $(-1, 1)$, with arbitrary positive scaling parameter α [32]. However, trying several different functions, we did not find any qualitative differences among the dynamics of different 'separability parameters' (provided that α is not too small). Thus we decided to confine ourselves to the function (8).

The determinants of the covariance and cross-covariance block matrices can be expressed as follows:

$$\begin{aligned} \det \mathcal{Q}_{kk} &= \langle \hat{x}_k^2\rangle\langle \hat{p}_k^2\rangle - \tfrac{1}{4}\langle \hat{x}_k\hat{p}_k + \hat{p}_k\hat{x}_k\rangle^2 \\ &= \tfrac{1}{4}\langle \hat{a}_k^\dagger\hat{a}_k + \hat{a}_k\hat{a}_k^\dagger\rangle^2 - |\langle \hat{a}_k^2\rangle|^2 \quad (9) \\ \det \mathcal{Q}_{kj} &= \langle \hat{x}_k\hat{x}_j\rangle\langle \hat{p}_k\hat{p}_j\rangle - \langle \hat{x}_k\hat{p}_j\rangle\langle \hat{p}_k\hat{x}_j\rangle \\ &= |\langle \hat{a}_k\hat{a}_j^\dagger\rangle|^2 - |\langle \hat{a}_k\hat{a}_j\rangle|^2. \quad (10) \end{aligned}$$

It is known that the separability of Gaussian states is determined mainly by the sign of the determinant $\det \mathcal{Q}_{kj}$ [30] (it is the only determinant which is sensitive to the so-called 'partial transposition', i.e., the change of sign of the momentum operator in one mode without an analogous change in another): the state is always separable (i.e., correlations between different modes are 'classical') if $\det \mathcal{Q}_{kj} > 0$. Suppose that coordinate–momentum correlations are absent, i.e., $\langle \hat{p}_k\hat{x}_j\rangle = 0$ (this happens in many special cases, when the density matrix is *real*). Then formula (10) gives an interesting interpretation of the physical meaning of entanglement for continuous variable systems: a necessary condition for the state to be entangled (i.e., to possess 'nonclassical' correlations) is *opposite signs of the coordinate–coordinate and momentum–momentum covariances*. If these signs coincide, the correlations are 'classical' (at least for Gaussian states).

It is worth noting also that the determinants of matrix $\mathcal{Q}$ and its diagonal blocks $\mathcal{Q}_{kk}$ are related to the *purities* of the coupled system of two modes and each mode alone according to the relations [12]

$$\mu \equiv \mathrm{Tr}\,\hat{\rho}^2 = (16 \det \mathcal{Q})^{-1/2} \quad (11)$$

$$\mu_k \equiv \mathrm{Tr}\,\hat{\rho}_k^2 = (4 \det \mathcal{Q}_{kk})^{-1/2}. \quad (12)$$

For example, for mixed factorized (i.e., separable) states of two modes we have $\mu = \mu_k\mu_j$, $\det \mathcal{Q}_{kj} = 0$, so

$$\mathcal{S}_{kj} = \tanh\left[\frac{(1-\mu_k^2)(1-\mu_j^2)}{16\mu_k^2\mu_j^2}\right]$$

and $\mathcal{S}_{kj} \to 1$ when at least one of the states has a very small value of the purity (thus being 'highly classical') and another state is mixed (i.e., its purity is less than 1).

In the following sections we analyse the dynamics of the separability coefficient (8) for different modes of the one-dimensional ideal cavity with an oscillating boundary, considering some examples of *Gaussian* initial states, which include as special cases *thermal* and *squeezed* states. It is known that initial Gaussian states remain Gaussian if the evolution is governed by quadratic Hamiltonians [12, 13]. Consequently, as soon as the problem is reduced to a Hamiltonian of the form (1), we can apply all formulae of this section. The only thing that we have to do is to calculate the time dependence of the elements of the variance matrices.

3. A quantum field in a 1D cavity with an oscillating boundary

We consider the model of a 1D cavity formed by two infinite ideal plates, whose positions are given by $x_{\text{left}} \equiv 0$ and $x_{\text{right}} \equiv L(t) = L_0(1 + \varepsilon \sin[p\omega_1 t])$ with $|\varepsilon| \ll 1$ and $p = 1, 2, \ldots$ ($\omega_1 = \pi c/L_0$). We suppose that the only component of the operator vector potential of the electromagnetic field $\hat{A}(x, t)$ is parallel to the plates (TE mode), thus obeying the Dirichlet boundary conditions. Then $\hat{A}(x, t)$ can be expanded in the Heisenberg representation over *initial* bosonic lowering/raising operators $\hat{b}_n$ and $\hat{b}_n^\dagger$ (satisfying the commutation relations $[\hat{b}_n, \hat{b}_k^\dagger] = \delta_{nk}$) as

$$\hat{A}(x, t) = \sum_{n=1}^{\infty} \frac{2}{\sqrt{n}}[\hat{b}_n\psi^{(n)}(x, t) + \text{h.c.}]. \quad (13)$$

Following [5], we look for the mode functions $\psi^{(n)}(x, t)$ in the form

$$\psi^{(n)}(x, t) = \sqrt{\frac{L_0}{L(t)}} \sum_{k=1}^{\infty} \sin\left[\frac{\pi k x}{L(t)}\right]\{\rho_k^{(n)}\mathrm{e}^{-\mathrm{i}\omega_k t} - \rho_{-k}^{(n)}\mathrm{e}^{\mathrm{i}\omega_k t}\} \quad (14)$$

where $\omega_k = k\omega_1$ and coefficients $\rho_{\pm k}^{(n)}$ depend on the slowly changing variable

$$\tau = \tfrac{1}{2}\varepsilon\omega_1 t. \quad (15)$$

The coefficients $\rho_k^{(n)}(\tau)$ satisfy an infinite system of coupled equations ($k = \pm 1, \pm 2, \ldots; n = 1, 2, \ldots$)

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\rho_k^{(n)} = (-1)^p[(k+p)\rho_{k+p}^{(n)} - (k-p)\rho_{k-p}^{(n)}] \quad (16)$$

which was solved in [5]. Here we confine ourselves to the simplest special case of *strict* resonance between the field modes and the motion of the boundary. Due to the initial conditions $\rho_k^{(n)}(0) = \delta_{kn}$, the solutions to (16) satisfy the relation $\rho_{j+mp}^{(k+np)} \equiv 0$ if $j \neq k$ (where $j, k = 0, 1, \ldots, p-1$). The nonzero coefficients read [5]

$$\begin{aligned} \rho_{j+mp}^{(j+np)}(\tau) &= \frac{\Gamma(1+n+j/p)(\sigma\kappa)^{n-m}}{\Gamma(1+m+j/p)\Gamma(1+n-m)} \\ &\quad \times F(n+j/p, -m-j/p; 1+n-m; \kappa^2) \quad (17) \end{aligned}$$

where

$$\kappa = \tanh(p\tau), \qquad \sigma \equiv (-1)^p, \tag{18}$$

and $F(a, b; c; z)$ is the Gauss hypergeometric function. The functions (17) are *exact* solutions to the set of equations (16), relating coefficients with different lower indices. Besides that, these functions satisfy another set of equations, which can be treated as recurrence relations with respect to the *upper* indices [5]:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\rho_m^{(n)} = n\sigma[\rho_m^{(n-p)} - \rho_m^{(n+p)}] \qquad n \geqslant p \quad \rho_m^{(0)} \equiv 0 \tag{19}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\rho_m^{(n)} = n\sigma[\rho_{-m}^{(p-n)*} - \rho_m^{(p+n)}] \qquad n = 1, 2, \ldots, p-1. \tag{20}$$

The consequences of equations (16), (19) and (20) are the identities

$$\sum_{m=-\infty}^{\infty} m\rho_m^{(n)*}\rho_m^{(k)} = n\delta_{nk} \tag{21}$$

$$\sum_{n=1}^{\infty} \frac{m}{n}[\rho_m^{(n)*}\rho_j^{(n)} - \rho_{-m}^{(n)*}\rho_{-j}^{(n)}] = \delta_{mj} \tag{22}$$

$$\sum_{n=1}^{\infty} \frac{1}{n}[\rho_m^{(n)*}\rho_{-j}^{(n)} - \rho_j^{(n)*}\rho_{-m}^{(n)}] = 0 \tag{23}$$

where *fixed* indices $(n, k$ or $m, j)$ are positive integers.

We suppose that after some interval of time T the moving wall comes back to its initial position L_0. Then for $t \geqslant T$ the field operator assumes the form

$$\hat{A}(x, t) = \sum_{m=1}^{\infty} \frac{2}{\sqrt{m}} \sin\left(\frac{\pi m x}{L_0}\right)[\hat{a}_m \mathrm{e}^{-\mathrm{i}\omega_m t} + \text{h.c.}] \tag{24}$$

where the 'final' operators $\hat{a}_m$ are related to the 'initial' operators $\hat{b}_n$ and $\hat{b}_n^\dagger$ by means of the Bogoliubov transformation ($\tau_T \equiv \frac{1}{2}\varepsilon\omega_1 T$):

$$\hat{a}_m = \sum_{n=1}^{\infty} \sqrt{\frac{m}{n}}[\hat{b}_n\rho_m^{(n)}(\tau_T) - \hat{b}_n^\dagger\rho_{-m}^{(n)*}(\tau_T)]. \tag{25}$$

The commutation relations $[\hat{a}_n, \hat{a}_k^\dagger] = \delta_{nk}$ hold due to the identities (21)–(23), which are nothing but the *unitarity conditions* of the transformation (25). These commutation relations, together with the expression for the energy of the field

$$\hat{H} \equiv \frac{1}{8\pi}\int_0^{L_0} \mathrm{d}x\, [(\partial\hat{A}/\partial t)^2 + (\partial\hat{A}/\partial x)^2] = \sum_{n=1}^{\infty} \omega_n(\hat{a}_n^\dagger\hat{a}_n + 1/2), \tag{26}$$

convince us that $\hat{a}_n$ and $\hat{a}_n^\dagger$ are true lowering and raising operators at $t \geqslant T$ (whereas at $t < 0$ the 'physical' operators were $\hat{b}_n$ and $\hat{b}_n^\dagger$).

4. Covariance matrices of field modes and the dynamics of the separability parameter in the 'principal resonance' case

The second-order moments of the operators $\hat{a}_n$ and $\hat{a}_n^\dagger$ can be calculated with the aid of equation (25), provided that one knows the initial second-order moments of the operators $\hat{b}_n$ and $\hat{b}_n^\dagger$. We suppose that the initial state of the field is factorized with respect to different modes, so that $\langle\hat{b}_r^\dagger\hat{b}_s\rangle = \langle\hat{b}_r\hat{b}_s\rangle = 0$ for any $r \neq s$. Moreover, we assume in this section that $\langle\hat{b}_r^2\rangle = 0$ for any r as well (i.e., that the initial states are diagonal in the Fock basis). Thus we exclude, e.g., initial *squeezed* states, but we admit the possibility of initial *thermal* states (which are Gaussian). Then the nonzero initial second-order moments are $\langle\hat{b}_r^\dagger\hat{b}_r\rangle = \nu_r$, where ν_r is the initial mean number of photons in the rth mode. In such a case, it is convenient to split the second-order moments of the operators $\hat{a}_n$ and $\hat{a}_n^\dagger$ into 'vacuum' and 'nonvacuum' parts:

$$\langle\hat{a}_r\hat{a}_s\rangle = \langle\hat{a}_r\hat{a}_s\rangle^{(\mathrm{v})} + \langle\hat{a}_r\hat{a}_s\rangle^{(\mathrm{nv})},$$

$$\langle\hat{a}_r^\dagger\hat{a}_s\rangle = \langle\hat{a}_r^\dagger\hat{a}_s\rangle^{(\mathrm{v})} + \langle\hat{a}_r^\dagger\hat{a}_s\rangle^{(\mathrm{nv})}$$

with

$$\langle\hat{a}_r\hat{a}_s\rangle^{(\mathrm{v})} = -\sum_{n=1}^{\infty}\frac{\sqrt{rs}}{n}\rho_r^{(n)}\rho_{-s}^{(n)*} \tag{27}$$

$$\langle\hat{a}_r^\dagger\hat{a}_s\rangle^{(\mathrm{v})} = \sum_{n=1}^{\infty}\frac{\sqrt{rs}}{n}\rho_{-r}^{(n)}\rho_{-s}^{(n)*} \tag{28}$$

$$\langle\hat{a}_r\hat{a}_s\rangle^{(\mathrm{nv})} = -\sqrt{rs}\sum_{j=1}^{\infty}\frac{\nu_j}{j}[\rho_r^{(j)}\rho_{-s}^{(j)*} + \rho_{-r}^{(j)*}\rho_s^{(j)}] \tag{29}$$

$$\langle\hat{a}_r^\dagger\hat{a}_s\rangle^{(\mathrm{nv})} = \sqrt{rs}\sum_{j=1}^{\infty}\frac{\nu_j}{j}[\rho_{-r}^{(j)}\rho_{-s}^{(j)*} + \rho_r^{(j)*}\rho_s^{(j)}]. \tag{30}$$

The coefficients $\rho_{\pm m}^{(n)}$ in equations (27)–(30) should be taken at the moment T; thus their argument is τ_T. Strictly speaking, these equations have physical meanings at those moments of time T when the wall returns to its initial position, i.e. for $T = N\pi/p$ with N an integer. Consequently, the argument τ_T of the coefficients $\rho_{\pm m}^{(n)}$ in (27)–(30) assumes discrete values $\tau^{(N)} = N\varepsilon\pi/(2p)$. One should have in mind, however, that an interesting situation arises in our problem for the values $\tau \sim 1$ (or larger). Then $N \sim \varepsilon^{-1} \gg 1$, and the minimal increment $\Delta\tau \sim \varepsilon$ is so small that τ_T can be considered as a continuous variable (under realistic conditions, $\varepsilon \leqslant 10^{-8}$ [2]). For this reason, we omit the subscript T, writing simply τ instead of τ_T or $\tau^{(N)}$.

In the special 'semiresonance' case $p = 1$, the only possible value of the index j in equation (17) is $j = 0$, and all coefficients $\rho_m^{(n)}$ with *negative* lower index $m = -1, -2, \ldots$ go to zero, due to the poles of the Gamma function $\Gamma(1 + m)$ in (17). Thus all 'vacuum' second-order statistical moments of the creation and annihilation operators turn into zero for $p = 1$ (in particular, there is no photon generation from the vacuum in the 'semiresonance' case [27, 28]). Moreover, for nonvacuum factorized and 'diagonal' initial states (i.e., without off-diagonal matrix elements in the Fock basis), the only nonzero moments are $\langle\hat{a}_r^\dagger\hat{a}_s\rangle$. Then equation (10) shows that the determinant $\det\mathcal{Q}_{kj}$ is always *nonnegative*. This means, according to [30], that any two modes remain *separable* forever if they were initially in a separable Gaussian thermal state. The initial vacuum state (with $\det\mathcal{Q}_{kj} = 0$) does not evolve with time for $p = 1$, also remaining separable.

Consequently, for initial thermal states, inseparability can arise only for $p \geqslant 2$. We confine ourselves here to the case of 'principal' [5] resonance with $p = 2$. To calculate the 'vacuum' sums, we differentiate the equations (27) and (28) with respect to the 'slow time' τ. Then, taking into account the recurrence relations (19) and (20), one can verify that the fraction $1/n$ is cancelled. After that, if necessary changing the summation index n to $n \pm p$, one can verify that almost all terms in the right-hand sides are cancelled, and the infinite series are reduced to finite sums (for details see [5, 17]). Taking into account that all functions $\rho_m^{(n)}$ are *real*, according to equation (17) (this is because we consider here the case of *strict resonance*), we finally arrive at the equations

$$\mathrm{d}\langle \hat{a}_r \hat{a}_s\rangle^{(\mathrm{v})}/\mathrm{d}\tau = -\sqrt{rs}[\rho_r^{(1)}\rho_s^{(1)} + \rho_{-r}^{(1)}\rho_{-s}^{(1)}] \qquad (31)$$

$$\mathrm{d}\langle \hat{a}_r^\dagger \hat{a}_s\rangle^{(\mathrm{v})}/\mathrm{d}\tau = \sqrt{rs}[\rho_r^{(1)}\rho_{-s}^{(1)} + \rho_{-r}^{(1)}\rho_s^{(1)}]. \qquad (32)$$

For $p = 2$, only odd modes can be excited from the initial vacuum state. Moreover, the hypergeometric functions in the formula (17) for coefficients $\rho_r^{(n)}$ with $j = 1$ are reduced in this case to some combinations of the complete elliptic integrals of the first and the second kinds [5] (see examples in appendix A), so equations (31) and (32) can be integrated for any values of the indices r and s; see [5, 17] or appendix B for technical details. The explicit forms of the 'vacuum parts' of the second-order moments with suffixes $r = 1, 3$ are given in appendix C.

In a generic case, the determinant of the symmetrical 4×4 covariance matrix $\mathcal{Q}$ (4) contains 17 different terms (see its explicit form in, e.g., [33]). However, in the specific case of *strict resonance* considered here, all covariances between the 'coordinate' and 'momenta' operators turn out to be equal to zero identically: $\widetilde{x_k p_j} = 0$ (because all Bogoliubov's coefficients $\rho_m^{(n)}$ are real), and for this reason the determinant of the covariance matrix for the ith and jth modes can be factorized as

$$\det \mathcal{Q} = (\widetilde{p_i p_i}\,\widetilde{p_j p_j} - \widetilde{p_i p_j}^2)(\widetilde{x_i x_i}\,\widetilde{x_j x_j} - \widetilde{x_i x_j}^2) \qquad (33)$$

where nonzero covariances are given by the formulae

$$\widetilde{x_i x_j} = [\langle \hat{a}_i^\dagger \hat{a}_j\rangle + \tfrac{1}{2}\delta_{ij} + \mathrm{Re}\langle \hat{a}_i \hat{a}_j\rangle](\omega_i\omega_j)^{-1/2} \qquad (34)$$

$$\widetilde{p_i p_j} = [\langle \hat{a}_i^\dagger \hat{a}_j\rangle + \tfrac{1}{2}\delta_{ij} - \mathrm{Re}\langle \hat{a}_i \hat{a}_j\rangle](\omega_i\omega_j)^{1/2}. \qquad (35)$$

In the case involved the symbols for the real parts in (34) and (35) can be omitted, as soon as all coefficients $\rho_m^{(n)}$ are real. It is worth mentioning that the scaling factors ω_k in the definition of the quadrature components (2) do not influence the values of the determinants in formulae (9), (10) and (33), and consequently the value of the separability coefficient (8).

Now we notice that asymptotically, as $\tau \to \infty$, the Bogoliubov coefficients depend neither on the upper indices nor on the signs of the lower indices [17]:

$$\rho_{2m+1}^{(2n+1)}(\infty) = \rho_{-2m-1}^{(2n+1)}(\infty) = \frac{2(-1)^m}{\pi(2m+1)}. \qquad (36)$$

Then, looking at equations (29)–(32), one can conclude that the second-order moments of raising and lowering operators have the following form for $\tau \gg 1$:

$$\langle \hat{a}_r \hat{a}_s\rangle = -\frac{8(-1)^{(r-s)/2}}{\pi^2\sqrt{rs}}(\tau + \mathcal{Z} + f_{rs}) \qquad (37)$$

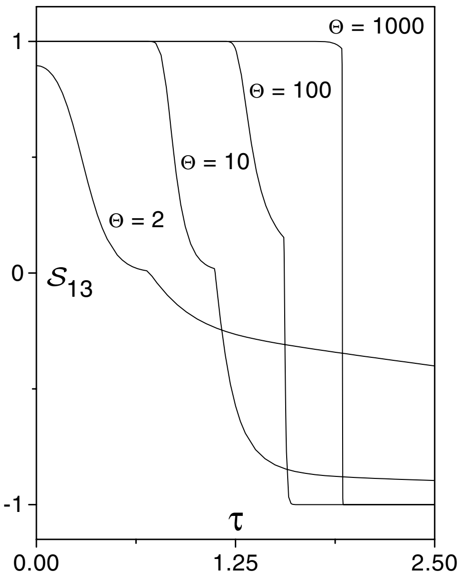

Figure 1. The dependence of the coefficient of separability, $\mathcal{S}_{13}$, between the first and third modes on the 'slow time' τ in the principal resonance case $p = 2$, for different initial dimensionless temperatures $\Theta = 2, 10, 100, 1000$.

$$\langle \hat{a}_r^\dagger \hat{a}_s\rangle = \frac{8(-1)^{(r-s)/2}}{\pi^2\sqrt{rs}}(\tau + \mathcal{Z} + g_{rs}) \qquad (38)$$

where r and s are *odd* numbers and

$$\mathcal{Z} = \sum_{k=0}^{\infty} \frac{\nu_{2k+1}}{2k+1}. \qquad (39)$$

The coefficients f_{rs} and g_{rs} are of the order of unity (compare with $\tau \gg 1$), and they do not depend on the initial distribution of photons $\{\nu_k\}$, since they can be obtained by integrating 'vacuum' equations (31) and (32), respectively. In view of (34), (35), (37), and (38), all determinants of covariance matrices, (9), (10), and (33), are proportional to the combination $\tau + \mathcal{Z}$ for $\tau \gg 1$. Consequently, the separability parameter $\mathcal{A}_{rs}$ (7) is also proportional to $(\tau + \mathcal{Z})$ for $\tau \gg 1$. This means that the sign of $\mathcal{A}_{rs}$ (and $\mathcal{S}_{rs}$) for $\tau \gg 1$ does not depend on the initial state of the field mode: it is the same as in the case of the initial vacuum state. It appears that this sign is negative for any pair of indices r, s. Consequently, although the joint state of any two modes becomes highly mixed as $\tau \to \infty$ (see equations (11) and (12)), this state is asymptotically *maximally inseparable*: $\mathcal{S}_{rs} \to -1$ for any initial state. On the other hand, the additional term $\mathcal{Z}$ 'enhances' the effect, in the sense that if the quantity $\mathcal{A}_{rs}$ is negative (and large in its absolute value) for the vacuum state, it becomes even 'more negative' with increase of $\mathcal{Z}$, i.e., with increase of the temperature for the initial thermal states. For this reason, the dependence of the separability coefficient $\mathcal{S}_{rs}$ on time τ becomes 'sharper' with increase of the temperature. This counterintuitive behaviour (usually, increasing temperature is accompanied by 'smoothing' effects) is shown in figure 1. We consider initial equilibrium thermal states, assuming that the mean number of photons in the jth mode is given by the Planck distribution

$$\nu_j = [\exp(j/\Theta) - 1]^{-1}, \qquad (40)$$

Θ being a dimensionless temperature. Then $\mathcal{Z} \approx \Theta$ for $\Theta \gg 1$ and $\mathcal{Z} \approx \exp(-1/\Theta)$ for $\Theta \ll 1$. Jumps of the time derivatives of the curves in figure 1 near the points of transition from the 'classical' to the 'quantum' regime of correlations are due to the discontinuity of the derivative of the function $|\det \mathcal{Q}_{jk}|$ in equation (7).

A natural question arises: whether a transition from 'classical' to 'quantum' intermode correlations is accompanied by significant changes in behaviour of other physical quantities. In order to answer this question, we considered the behaviour of the quantity characterizing correlations between photon numbers of coupled modes:

$$\mathcal{K}_{jk} = \langle(\hat{N}_j - \langle\hat{N}_j\rangle)(\hat{N}_k - \langle\hat{N}_k\rangle)\rangle \equiv \langle\hat{N}_j\hat{N}_k\rangle - \langle\hat{N}_j\rangle\langle\hat{N}_k\rangle \quad (41)$$

where $\hat{N}_k \equiv \hat{a}_k^\dagger\hat{a}_k$. In a generic case, $\mathcal{K}_{jk}$ depends on the moments of the first, second, third, and fourth orders of raising and lowering operators. However, for *Gaussian states* all higher order moments can be expressed in terms of the first- and second-order ones. For example, the fourth-order centralized moments of any two real commuting quadratures z_i and z_j (where each z_k may be either x_k or p_k) can be written as [12]

$$\langle(\delta\hat{z}_i)^2(\delta\hat{z}_j)^2\rangle = \langle(\delta\hat{z}_i)^2\rangle\langle(\delta\hat{z}_j)^2\rangle + 2\langle\delta\hat{z}_i\delta\hat{z}_j\rangle^2$$

where $\delta\hat{z}_k \equiv \hat{z}_k - \langle\hat{z}_k\rangle$. Thus for Gaussian states with zero mean values of the quadrature operators we have

$$\mathcal{K}_{jk} = |\langle\hat{a}_j\hat{a}_k\rangle|^2 + |\langle\hat{a}_j\hat{a}_k^\dagger\rangle|^2 \equiv \tfrac{1}{2}\,\mathrm{Tr}(\mathcal{Q}_{jk}\mathcal{Q}_{kj}), \quad (42)$$

so in this special case $\mathcal{K}_{jk}$ almost coincides (the difference is in normalization factors) with the *trace-covariance correlation coefficient* introduced (on other grounds) in [34]. The time evolution of this coefficient for different initial states and different quantum systems was considered also in [35, 36]. In figure 2 we compare the time dependence of the normalized coefficient $\tilde{\mathcal{K}}_{13} = \mathcal{K}_{13}/(\nu_1\nu_3)$ for the first and third modes with the evolution of the separability coefficient $\mathcal{S}_{13}$ under the same conditions. One can see that nothing happens with the coefficient $\tilde{\mathcal{K}}_{13}$ when a separable state of two modes is transformed to an inseparable one (similar plots were made for many other choices of mode indices j, k and temperature Θ). The time dependence of the mean photon number in each mode also does not show any change in its behaviour at the moment of transition from a separable to an inseparable state. These observations force us to think that either the mean numbers of photons in each mode and their correlation functions are not sensitive to entanglement at all, or possible relations between physical (energy, photon statistics, squeezing, etc) and geometric (separability, degree of nonclassicality, etc) properties of Gaussian quantum states are well hidden. Examples of other Hamiltonians of interactions between coupled modes, considered in [32], led to similar conclusions.

5. Evolution of entangled modes in the 'semiresonance' case

As was stated already in section 4, in the 'semiresonance' case $p = 1$ the Bogoliubov transformation (25) does not contain

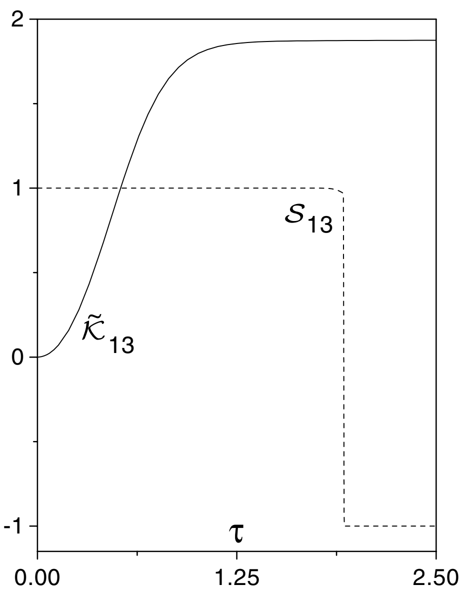

Figure 2. The normalized photon number coefficient of correlation, $\tilde{\mathcal{K}}_{13} = \mathcal{K}_{13}/(\nu_1\nu_3)$, between the first and third modes and the separability coefficient $\mathcal{S}_{13}$, as functions of the 'slow time' τ, in the principal resonance case $p = 2$, for the dimensionless initial temperature $\Theta = 1000$.

the terms with creation operators $\hat{b}_m^\dagger$. As a consequence, initial thermal states always remain separable, i.e., correlations between different modes have 'classical' nature. Different measures of such correlations were calculated in [36].

But what can happen if the initial state has been entangled? To answer this question, we consider below a special case, where initially the first two modes (with $k = 1$ and $k = 2$) are in the so-called *two-mode squeezed state* [37]

$$|\lambda\rangle = \sqrt{1-|\lambda|^2}\sum_{n=0}^{\infty}\lambda^n|n,n\rangle \qquad |\lambda| < 1 \quad (43)$$

(frequently considered as an example of 'maximally entangled' states), whereas all other modes with $k \geqslant 3$ are in the vacuum state at $t = 0$. For this *pure* Gaussian state, we have (in the equations below, $k = 1, 2$)

$$\det\mathcal{Q} = 1/16, \qquad \langle\hat{b}_1^\dagger\hat{b}_2\rangle = \langle\hat{b}_k^2\rangle = 0,$$

$$\langle\hat{b}_1\hat{b}_2\rangle = \frac{\lambda}{1-|\lambda|^2}, \qquad \langle\hat{b}_k^\dagger\hat{b}_k\rangle = \frac{|\lambda|^2}{1-|\lambda|^2},$$

$$\det\mathcal{Q}_{kk} = \frac{1}{4}\left(\frac{1+|\lambda|^2}{1-|\lambda|^2}\right)^2, \qquad \det\mathcal{Q}_{12} = -\frac{|\lambda|^2}{(1-|\lambda|^2)^2},$$

so the separability coefficient

$$\mathcal{S}_{12} = -\tanh\left[\frac{|\lambda|^2}{(1-|\lambda|^2)^2}\right]$$

tends to -1 when $|\lambda| \to 1$.

For an arbitrary initial state, the evolution of the second-order moments in the case $p = 1$ is given by the formulae

$$\langle\hat{a}_r\hat{a}_s\rangle = \sum_{n,k=1}^{\infty}\sqrt{\frac{rs}{nk}}\langle\hat{b}_n\hat{b}_k\rangle\rho_r^{(n)}\rho_s^{(k)}$$

$$\langle \hat{a}_r^\dagger \hat{a}_s \rangle = \sum_{n,k=1}^{\infty} \sqrt{\frac{rs}{nk}} \langle \hat{b}_n^\dagger \hat{b}_k \rangle \rho_r^{(n)*} \rho_s^{(k)}.$$

However, due to the chosen initial conditions, the sums are reduced to a few terms, containing only real coefficients $\rho_r^{(n)}$ with $n = 1, 2$, whose explicit form is given in appendix A. After some calculations we arrive at the following expression for the coefficient $\mathcal{A}_{12}$ (7) as a function of the variables $\kappa = \tanh\tau$ and $\tilde{\kappa} = \sqrt{1-\kappa^2} = 1/\cosh\tau$ (we assume λ in (43) to be real parameter):

$$\mathcal{A}_{12}(\kappa) = \frac{\lambda^2 \tilde{\kappa}^8 [D - |D| + 2\lambda^2 \tilde{\kappa}^4 \kappa^8 (6\tilde{\kappa}^2 + \kappa^4)]}{2(1-\lambda^2)^2}, \quad (44)$$

where

$$D = 18\lambda^2\kappa^6 - (1 - 5\kappa^2)^2. \quad (45)$$

The sign of the separability coefficient is determined mainly by the sign of the parameter D (which coincides with the sign of $\det \mathcal{Q}_{12}$), because $\mathcal{A}_{12}$ is automatically positive for $D > 0$, as is obvious from equation (44). But for any $\lambda \neq 0$, $D(\kappa) > 0$ in the vicinity of point $\kappa^2 = 1/5$ (or $\tau = \ln[(1+\sqrt{5})/2] \approx 0.48$). Therefore, at least during some time interval, the joint state of two (initially inseparable) modes becomes *separable*. Moreover, if $\lambda^2 > 8/9 \approx 0.89$, then D becomes positive as $\kappa \to 1$, which results in the asymptotical positivity of $\mathcal{A}_{12}$ and separability of the modes.

For arbitrary modes with numbers r and s we obtain

$$\det \mathcal{Q}_{rs} = \frac{rs\lambda^2\tilde{\kappa}^8\kappa^{2(r+s-4)}}{4(1-\lambda^2)^2} [\lambda^2 \alpha_{rs}^2(\kappa) - 2\kappa^2 \beta_{rs}^2(\kappa)]$$

with

$$\alpha_{rs}(\kappa) = (r-1)(s-1) - 2(rs-2)\kappa^2 + (r+1)(s+1)\kappa^4$$

$$\beta_{rs}(\kappa) = r + s - 2 - (r+s+2)\kappa^2.$$

For $\kappa \ll 1$, $\det \mathcal{Q}_{rs}$ is negative if $r = 1$ (or $s = 1$), but it is positive for $r, s > 1$. Also, it is always positive in the vicinity of the point $\kappa_*^2 = (r+s-2)/(r+s+2)$. However, the sign of $\det \mathcal{Q}_{rs}$ at $\kappa \to 1$ does not depend on r and s, being determined by the sign of the combination $9\lambda^2 - 8$. This means that all modes are entangled asymptotically in the case of initial two-mode squeezed states with not very big values of the parameter λ, but their states go, asymptotically, to separable mixtures (with respect to any pair of modes) for highly entangled initial states with $\lambda^2 > \lambda_c^2 = 8/9$ (i.e., when the first and second modes have initially more than eight photons)—a quite unexpected and even counterintuitive result!

In figure 3 we show time dependences of the coefficient of separability between the first and second modes for different values of the real parameter λ. The values $\lambda = 0.94$ and 0.6 are less than the critical value $\lambda_c \approx 0.942\,81$, so the corresponding curves tend asymptotically to zero, remaining *negative* for $\tau > 1$. This means that the states with these values of λ are inseparable asymptotically, although the degree of inseparability can be very small. If λ only slightly exceeds the critical value, then the modes become separable for $\tilde{\kappa}^2 < 9(\lambda^2 - \lambda_c^2)/4$. In this case, the plot of $\mathcal{S}_{12}(\tau)$ differs from the solid curve in figure 3 by having a small positive bump at some big value of τ.

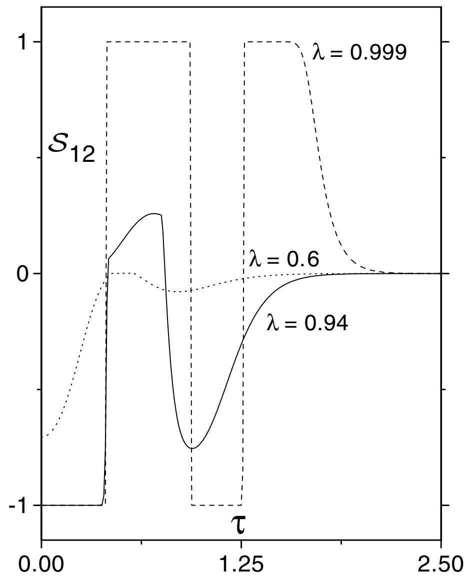

Figure 3. The coefficient of separability, $\mathcal{S}_{12}$, between the first and second modes in the 'semiresonance' case $p = 1$ for the initial two-mode squeezed state (43) with different values of the real parameter λ.

For $\lambda = 1$ the coefficient $D(\kappa)$ defined by (45) turns into zero for $\kappa_1 = (\sqrt{7}-1)/3\sqrt{2} \approx 0.39$, $\kappa_2 = 1/\sqrt{2} \approx 0.71$, and $\kappa_3 = (\sqrt{7}+1)/3\sqrt{2} \approx 0.86$, which correspond to the values $\tau_1 \approx 0.41$, $\tau_2 \approx 0.89$, and $\tau_3 \approx 1.29$. Real moments of transition from separable to inseparable states are slightly shifted from these values due to the presence of the last (positive) term inside the square brackets in (44), as one can see from figure 3. For instance, the second transition happens at $\kappa \approx 0.73$ and $\tau \approx 0.93$, if $1 - \lambda \ll 1$. Approximate asymptotical dependences $\mathcal{S}_{12}(\tau)$ for $\tau \gg 1$ are as follows:

$$\mathcal{S}_{12}(\tau) \approx \begin{cases} \dfrac{9\lambda^2(\lambda^2 - \lambda_c^2)}{(1-\lambda^2)^2\cosh^8(\tau)}, & \lambda < \lambda_c \\ \dfrac{\lambda^4}{(1-\lambda^2)^2\cosh^{12}(\tau)}, & \lambda > \lambda_c. \end{cases}$$

In figure 4 we compare time dependences of the normalized photon number coefficient of correlation, $\mathcal{K}_{12}(\tau)/\mathcal{K}_{12}(0)$, between the first and second modes, normalized mean numbers of photons in the same modes, $\tilde{\mathcal{N}_k} \equiv \mathcal{N}_k(\tau)/\mathcal{N}_k(0)$, $k = 1, 2$, and the separability coefficient $\mathcal{S}_{12}$ for $\lambda = 0.999$. Again, as in the case $p = 2$, we do not see any correlations in the behaviour of the separability coefficient and statistical properties of the photon number distributions of the modes.

We can consider also some other statistical properties of the quantum states. For example, Mandel's Q-parameter of the first mode equals

$$Q_1 = \frac{\tilde{\kappa}^4(1 + 12\kappa^2 + 4\kappa^4)}{(1-\lambda^2)(1+2\kappa^2)} - 1.$$

Being strongly positive initially ($Q_1 = \lambda^2/(1-\lambda^2)$ for $\kappa = 0$), it goes to -1 when $\kappa \to 1$. For $\lambda = 0.999$, the transition from super-Poissonian to sub-Poissonian statistics happens when $\tilde{\kappa} \approx 0.137$, i.e., $\tau \approx 2.7$. At this moment of time, the state had

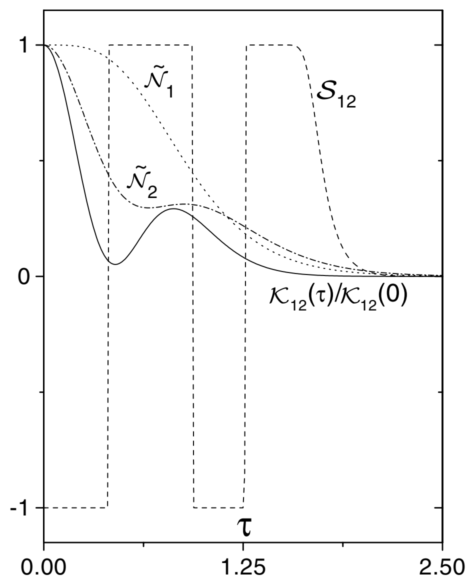

Figure 4. The normalized photon number coefficient of correlation, $\mathcal{K}_{12}(\tau)/\mathcal{K}_{12}(0)$, between the first and second modes, normalized mean numbers of photons in the same modes, $\tilde{\mathcal{N}}_k \equiv \mathcal{N}_k(\tau)/\mathcal{N}_k(0)$, $k = 1, 2$, and the separability coefficient $\mathcal{S}_{12}$, as functions of the 'slow time' τ, in the semiresonance case $p = 1$, for the initial two-mode squeezed state (43) with $\lambda = 0.999$.

become separable long ago, and nothing significant happens to it from the point of view of entanglement, according to figure 3.

Another important physical characteristic of a quantum state is the minimal value of the variances of the quadrature components of the family of operators $\hat{a}e^{i\gamma}$ with respect to the parameter γ in the interval $0 \leqslant \gamma < 2\pi$, known under the names 'principal squeezing' [38] and 'invariant squeezing' [39]. It is given by the formula (for the states with $\langle\hat{a}\rangle = 0$, which holds for all examples considered in this paper)

$$\sigma_{\min} = \tfrac{1}{2} + \langle\hat{a}^\dagger\hat{a}\rangle - |\langle\hat{a}^2\rangle|. \qquad (46)$$

For the first mode we obtain the expression

$$\sigma_{\min}^{(1)} = \frac{1}{2} + \frac{\lambda\tilde{\kappa}^4}{1-\lambda^2}[(1-\sqrt{2}\kappa)^2 - (1-\lambda)(1+2\kappa^2)]. \qquad (47)$$

The initial ($\kappa = 0$) states of the first and second modes are not squeezed (moreover, they are 'strongly unsqueezed', i.e., the minimal quadrature variances are much bigger than 1/2, if $1-\lambda \ll 1$), despite the name given to the state (43) (squeezing in this state is achieved for some quadrature component which is a linear combination of quadratures of two modes). But in the process of evolution the first mode becomes squeezed ($\sigma_{\min} < 1/2$), when variable κ belongs to the interval (κ_-, κ_m), where $\kappa_m = \min(1, \kappa_+)$ and $\kappa_\pm = (1 \pm \sqrt{1-\lambda^2})/(\lambda\sqrt{2})$ (note that $\kappa_- < 1$ for any $\lambda < 1$). Here we meet again the critical parameter $\lambda_c = \sqrt{8/9}$: if $\lambda < \lambda_c$, then the state remains squeezed for any $\tau > \tanh^{-1}(\kappa_-)$ (although the degree of squeezing $\sigma_{\min} - 1/2$ tends to zero as $\tau \to \infty$), whereas for $\lambda > \lambda_c$ the state is squeezed only during some finite interval of time. However, no evident correlations between the dynamics of squeezing and inseparability are observed. For example, if $\lambda = \lambda_c$, then $\kappa_- = 1/2$. But for these values of λ and κ we obtain $D = 3/16 > 0$, so the transition to the squeezed state of the first mode takes place when the two modes are well separable. The maximal absolute value of the (negative) difference $\sigma_{\min} - 1/2$ is close to 0.12, and it is achieved for $\kappa = (1 + \sqrt{17})/8 \approx 0.64$. For these values of λ and κ, $D = 0$ exactly, so this moment corresponds to the point of the maximum (with a discontinuity of the derivative) of the solid curve in figure 3 (when the state is still separable). After that, the modes become inseparable, but a (small) squeezing effect remains for all moments of time. On the other hand, if λ is very close to 1, then the time interval for which the state is squeezed is very short ($\Delta\tau \approx 4\sqrt{1-\lambda}$), but in the middle of this interval (at $\kappa = (\lambda\sqrt{2})^{-1}$) we obtain an appreciable squeezing: $\sigma_{\min} \approx 1/4$. This also happens when the two modes are separable (inside the first dashed rectangle in figure 3, although close to its right vertical side), but now, squeezing disappears before the joint state of the first and second modes becomes inseparable.

6. Conclusion

We have considered the separability of quantum states of coupled modes of the massless scalar field in a one-dimensional Fabry–Perot cavity with an oscillating boundary, whose frequency ω_w is in resonance with the frequency of the fundamental field mode ω_1. We have demonstrated a qualitative difference between the cases $\omega_w = \omega_1$ ('semiresonance') and $\omega_w = 2\omega_1$ ('principal resonance'). In the first case, any initial factorized quantum state remains separable forever, while the dynamics of separability of initially entangled states turns out to be more complicated. In particular, there always exist finite intervals of time when the joint mixed state of two modes becomes separable. Moreover, only the states with a not very high initial degree of entanglement remain entangled asymptotically, whereas strongly entangled initial two-mode squeezed states inevitably become separable statistical mixtures. In contrast, in the principal resonance case any initial factorized quantum state goes asymptotically to a maximally inseparable mixed state.

It is still unclear whether there exist incontestable correlations between the dynamics of separability and photon statistics in the field modes. We did not find such correlations in various examples considered. On the other hand, we discovered that some critical parameters, distinguishing different regimes of separability and squeezing (which are rather different phenomena), coincide. We may conclude that systems with moving boundaries seem to be good 'theoretical laboratories' for studying the problems of entanglement and decoherence. Therefore, it is worth continuing the studies begun in [24–26, 36] in these directions. Perhaps different (multi)photon distribution functions and off-diagonal matrix elements of Gaussian states in the Fock basis could give some additional information on the subject.

Acknowledgments

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Appendix A. Some explicit Bogoliubov coefficients

In the case of a strict 'principal' resonance ($p = 2$), the nonzero coefficients $\rho_m^{(n)}$ with $m = \pm 1, \pm 3$ and $n = 1, 3$ depend on $\kappa = \tanh(2\tau)$ and $\tilde{\kappa} \equiv \sqrt{1-\kappa^2} = [\cosh(2\tau)]^{-1}$ as follows [5, 17]:

$$\rho_1^{(1)} = \frac{2}{\pi}\mathbf{E}(\kappa) \qquad \rho_{-1}^{(1)} = \frac{2}{\pi\kappa}[\mathbf{E}(\kappa) - \tilde{\kappa}^2\mathbf{K}(\kappa)] \tag{A.1}$$

$$\rho_3^{(1)} = \frac{2}{3\pi\kappa}[(1-2\kappa^2)\mathbf{E}(\kappa) - \tilde{\kappa}^2\mathbf{K}(\kappa)] \qquad \rho_{-3}^{(1)} = -\frac{2}{3\pi\kappa^2}[(2-\kappa^2)\mathbf{E}(\kappa) - 2\tilde{\kappa}^2\mathbf{K}(\kappa)] \tag{A.2}$$

$$\rho_1^{(3)} = \frac{2}{\pi\kappa}[(2\kappa^2-1)\mathbf{E}(\kappa) + \tilde{\kappa}^2\mathbf{K}(\kappa)] \qquad \rho_{-1}^{(3)} = \frac{2}{\pi\kappa^2}[(2-\kappa^2)\mathbf{E}(\kappa) - 2\tilde{\kappa}^2\mathbf{K}(\kappa)] \tag{A.3}$$

$$\rho_3^{(3)} = \frac{2}{3\pi\kappa^2}[(-8\kappa^4+7\kappa^2)\mathbf{E}(\kappa) - 4\kappa^2\tilde{\kappa}^2\mathbf{K}(\kappa)] \qquad \rho_{-3}^{(3)} = \frac{2}{3\pi\kappa^3}[(7\kappa^2-8)\mathbf{E}(\kappa) + (8-3\kappa^2)\tilde{\kappa}^2\mathbf{K}(\kappa)] \tag{A.4}$$

where

$$\mathbf{K}(\kappa) = \int_0^{\pi/2} \frac{\mathrm{d}\alpha}{\sqrt{1-\kappa^2\sin^2\alpha}} = \frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;\kappa^2\right)$$

$$\mathbf{E}(\kappa) = \int_0^{\pi/2} \mathrm{d}\alpha\sqrt{1-\kappa^2\sin^2\alpha} = \frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;\kappa^2\right)$$

are the complete elliptic integrals of the first and the second kinds.

In the 'semiresonance' case ($p = 1$) it is convenient to use the expression obtained from (17) by means of a transformation $z \to 1-z$ in the argument of the hypergeometric function [27] (now $\kappa = \tanh\tau$ and $\tilde{\kappa} = \sqrt{1-\kappa^2} = 1/\cosh\tau$):

$$\rho_m^{(n)} = n(-1)^{n-1}\tilde{\kappa}^2\kappa^{m-n}F(1-n, m+1; 2; \tilde{\kappa}^2).$$

In particular, for $n = 1, 2$ we have

$$\rho_m^{(1)} = \tilde{\kappa}^2\kappa^{m-1} \qquad \rho_m^{(2)} = -2\tilde{\kappa}^2\kappa^{m-2}\left(1 - \frac{m+1}{2}\tilde{\kappa}^2\right)$$

$$\rho_1^{(1)} = \tilde{\kappa}^2 \qquad \rho_2^{(1)} = \tilde{\kappa}^2\kappa$$

$$\rho_1^{(2)} = -2\tilde{\kappa}^2\kappa \qquad \rho_2^{(2)} = \tilde{\kappa}^2(1-3\kappa^2).$$

Other explicit examples can be found in [36].

Appendix B. Solving differential equations containing products of complete elliptic integrals

To calculate, for instance, the vacuum average value $\langle\hat{a}_1^\dagger\hat{a}_3\rangle^{(v)}$, we use equations (32) and (A.1)–(A.4), replacing the derivative over τ by the derivative with respect to κ in accordance with the relation $\mathrm{d}\kappa/\mathrm{d}\tau = 2\tilde{\kappa}^2$. In this way we arrive at the equation

$$\frac{\mathrm{d}\langle\hat{a}_1^\dagger\hat{a}_3\rangle^{(v)}}{\mathrm{d}\kappa} = \frac{2\sqrt{3}}{3\pi^2\kappa^2\tilde{\kappa}^2}[\tilde{\kappa}^4\mathbf{K}^2(\kappa) + 2\kappa^2\tilde{\kappa}^2\mathbf{E}(\kappa)\mathbf{K}(\kappa) - (1+\kappa^2)\mathbf{E}^2(\kappa)]. \tag{B.1}$$

Taking into account the differentiation rules [40]

$$\frac{\mathrm{d}\mathbf{K}(\kappa)}{\mathrm{d}\kappa} = \frac{\mathbf{E}(\kappa)}{\kappa\tilde{\kappa}^2} - \frac{\mathbf{K}(\kappa)}{\kappa} \qquad \frac{\mathrm{d}\mathbf{E}(\kappa)}{\mathrm{d}\kappa} = \frac{\mathbf{E}(\kappa) - \mathbf{K}(\kappa)}{\kappa}$$

we may suppose that the factor $\tilde{\kappa}^2$ in the denominator of the right-hand side of equation (B.1) comes from the derivative $\mathrm{d}\mathbf{K}/\mathrm{d}\kappa$. Thus it is natural to look for a solution in the form

$$\langle\hat{a}_1^\dagger\hat{a}_3\rangle^{(v)} = \frac{2\sqrt{3}}{3\pi^2\kappa}[A(\kappa)\mathbf{K}^2(\kappa) + B(\kappa)\mathbf{K}(\kappa)\mathbf{E}(\kappa) + C(\kappa)\mathbf{E}^2(\kappa)] \tag{B.2}$$

where $A(\kappa)$, $B(\kappa)$, and $C(\kappa)$ are some polynomials of κ. Putting the expression (B.2) into equation (B.1) we obtain a set of coupled equations for the coefficients of these polynomials, which can be resolved recursively. The equations for other second-order moments can be integrated in the same manner.

Appendix C. Examples of the 'vacuum' second-order moments

Here we give explicit expressions (obtained as explained in appendix B) for the 'vacuum' second-order statistical moments of the lowest modes with $n = 1, 3$ in the case of strict resonance with $p = 2$. The argument of the complete elliptic integrals $\mathbf{E}$ and $\mathbf{K}$ is $\kappa = \tanh(2\tau)$, whereas $\tilde{\kappa} \equiv \sqrt{1-\kappa^2} = [\cosh(2\tau)]^{-1}$:

$$\langle\hat{a}_1^2\rangle^{(v)} = \frac{2}{\pi^2\kappa}[\tilde{\kappa}^2\mathbf{K}^2 - 2\mathbf{E}\mathbf{K} + \mathbf{E}^2]$$

$$\langle\hat{a}_3^2\rangle^{(v)} = \frac{2}{9\pi^2\kappa^3}[\tilde{\kappa}^2(4-\kappa^2)\mathbf{K}^2 - 2(2\kappa^4 - 3\kappa^2 + 4)\mathbf{E}\mathbf{K} + (4\kappa^4 - \kappa^2 + 4)\mathbf{E}^2]$$

$$\langle\hat{a}_1\hat{a}_3\rangle^{(v)} = -\frac{2\sqrt{3}}{3\pi^2\kappa^2}[\tilde{\kappa}^2\mathbf{K}^2 - 2\mathbf{E}\mathbf{K} + (1+\kappa^2)\mathbf{E}^2]$$

$$\langle\hat{a}_1^\dagger\hat{a}_3\rangle^{(v)} = \frac{2\sqrt{3}}{3\pi^2\kappa}[\tilde{\kappa}^2\mathbf{K}^2 + 2(\kappa^2-2)\mathbf{E}\mathbf{K} + 3\mathbf{E}^2]$$

$$\mathcal{E}_1^{(v)} = \frac{2}{\pi^2}\mathbf{K}(2\mathbf{E} - \tilde{\kappa}^2\mathbf{K})$$

$$\mathcal{E}_3^{(v)} = \frac{2}{3\pi^2\kappa^2}[(3\kappa^2-2)\mathbf{K}(2\mathbf{E} - \tilde{\kappa}^2\mathbf{K}) + 2(1+\kappa^2)\mathbf{E}^2]$$

where $\mathcal{E}_r^{(v)} = \langle\hat{a}_r^\dagger\hat{a}_r\rangle^{(v)} + \frac{1}{2}$.

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