Laguerre moments and generalized functions

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Abstract
Here we explore the link between the moments of the Laguerre polynomials or Laguerre moments and the generalized functions (as the Dirac delta-function and its derivatives), presenting several interesting relations. A useful application is related to a procedure for calculating mean values in quantum optics that makes use of the so-called quasi-probabilities. One of them, the P-distribution, can be represented by a sum over Laguerre moments when the electromagnetic field is in a photon-number state. Consequently, the P-distribution can be expressed in terms of Dirac delta-function and derivatives. More specifically, we found a direct relation between P-distributions and the Laguerre factorial moments.

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1. Introduction

The probability finding, as a measurement outcome, n photons in the field state $\hat{\rho}$ is given by $\text{Tr}(\hat{\rho}\ket{n}\bra{n})$ ($\hat{\rho}$ is a traceclass operator and $\ket{n}$ is the eigenstate of the photon-number operator). So, in a field prepared in a coherent state $\ket{\alpha}$ ($\hat{\rho} = \ket{\alpha}\bra{\alpha}$), where $\alpha$ is a complex number and $|\alpha|^2$ is the intensity of the field, or the mean photon number, then $\text{Tr}(\hat{\rho}\ket{n}\bra{n}) = |\langle n|\alpha\rangle|^2$.

On the other hand, from a formal point of view the state $\ket{\alpha}$ is used to map a q-number operator $\hat{O}(a, a^\dagger)$ ($a$ and $a^\dagger$ are destruction and creation operators of photons with respect to the number state $\ket{n}$, n = 0, 1, 2, . . . ) to a c-number function. The trace operation $\text{Tr}(\hat{\rho}|\alpha\rangle\langle\alpha|) = \langle \alpha | \hat{\rho} | \alpha \rangle = Q_{\hat{\rho}}(\alpha, \alpha^\ast)$ defines the Husimi distribution, or Q-distribution, for state $\hat{\rho}$ (actually this is a map: $\hat{\rho} \Rightarrow Q_{\hat{\rho}}(\alpha, \alpha^\ast), a \rightarrow \alpha, a^\dagger \rightarrow \alpha^\ast$).

The mean value of an operator $\hat{O}(a, a^\dagger)$ can be written as

$$\text{Tr}(\hat{\rho} \hat{O}) = \int O(\alpha, \alpha^\ast) P_{\hat{\rho}}(\alpha, \alpha^\ast) \, d^2\alpha$$ (1)
where $O(\alpha, \alpha^*) = \langle \alpha | \hat{O} | \alpha \rangle$ (this is also a map: $\hat{O} \Rightarrow O(\alpha, \alpha^*)$) and $P_\beta(\alpha, \alpha^*)$ is the Glauber–Sudarshan or P-distribution, related to $Q_\beta(\alpha, \alpha^*)$ through

$$P_\beta(\alpha, \alpha^*) = \exp\left(-\frac{\partial^2}{\partial \alpha \partial \alpha^*}\right) Q_\beta(\alpha, \alpha^*).$$

The distributions $Q_\beta(\alpha, \alpha^*)$ and $P_\beta(\alpha, \alpha^*)$ are quasi-probabilities, the former is always a smooth and well-behaved function of its arguments while the latter, depending on the state $\hat{\rho}$, may be a regular function or a generalized function (GF) as is the case for $\hat{\rho} = |n\rangle \langle n|.$

For a field in state $\hat{\rho} = |n\rangle \langle n|$ the probability to find $n$ photons in a coherent state $|\alpha\rangle \langle \alpha|$ is the same as the Q-distribution for state $|n\rangle \langle n|$, being a Poisson distribution in variable $n$ with mean value $|\alpha|^2 [1],

$$Q_n(|\alpha|^2) = |\langle n|\alpha \rangle|^2 = \frac{\exp(-|\alpha|^2)|\alpha|^{2n}}{n!}$$

(2)
a smooth and well-behaved function of its argument. However, the P-distribution is quite singular, as was originally reported in the classical papers of Glauber [2] and Sudarshan [3] and more recently reviewed by Wünsche [4], who found new relations and representations for the P-distribution. Working on this same problem of representing the P-distribution, we obtained several results which we did not find in the current literature, relating the Laguerre moments and Laguerre factorial moments to GFs. We also derived a direct relation between the P-distribution and the Laguerre factorial moments. These results are reported in this paper.

We begin by reminding, with some examples, how the Dirac delta-function arises in mathematical physics [5–10]:

(I) Certain sequences of functions defined on $\mathbb{R}$, $f_n(x), n = 1, 2, 3, \ldots$ are well behaved (continuous with continuous derivatives to all orders) in a domain $\mathcal{I}$; however, they cease to exist as such when $n \to \infty$, acquiring meaning as a continuous linear functional $T_f \equiv \langle f, \phi \rangle = \int_\mathcal{I} f(x) \phi(x) \, dx$ that maps each continuous test function $\phi(x)$ ($\phi \in \Phi$, $\Phi$ is a linear vector space) onto a complex number. So, the functional denoted as $T_f$ (or simply $f$) is called the distribution or GF. For instance, the functions

$$\frac{n}{\sqrt{\pi}} e^{-n^2/x^2} \quad \frac{n}{\pi} \frac{1}{1 + n^2 x^2} \quad \frac{\sin nx}{\pi x}$$

(3)

although being continuous with continuous derivatives to all orders for any finite integer $n$, no longer exhibit this property in the limit $n \to \infty$, thus no longer belong to the class of regular functions. All the examples in (3) converge to the so-called Dirac delta-function $\delta$, in reality a GF, to be referred to as the Dirac distribution (DD)

$$\delta(x) = \lim_{n \to \infty} \frac{n}{\sqrt{\pi}} e^{-n^2/x^2} = \lim_{n \to \infty} \frac{n}{\pi} \frac{1}{1 + n^2 x^2} = \lim_{n \to \infty} \frac{\sin nx}{\pi x}$$

defined by the functional

$$\langle T_{\delta_n}, \phi \rangle = \lim_{n \to \infty} \int_\mathcal{I} \delta_n(x - x_0) \phi(x) \, dx = \phi(x_0)$$

where $\delta_n(x)$ stands for any one of the functions displayed in (3) and $\phi(x)$ is a test function.

(II) From Sturm–Liouville theory we know that a class of second-order differential equations accept, as solution, orthogonal polynomials $P_n(x)$ that form a complete set, meaning that any piecewise smooth and bounded function $f(x)$ defined on $\mathcal{I}$ ($x \in \mathcal{I}$) can be expanded in terms of the $P_n(x)$ (the weight function and normalization factors are included in it),

$$f(x) = \sum_n c_n P_n(x)$$
the coefficients are obtained by integration,
\[ c_n = \int_I f(x)P_n(x) \, dx \]
and
\[ \sum_{n=0}^{\infty} P_n(x)P_n(x') = \delta(x - x') \]
is the completeness property of the polynomials. If the point 0 is contained in I, then setting \( x' = 0 \) in the previous equation, the DD becomes equal to an infinite weighted sum of polynomials,
\[ \sum_{n=0}^{\infty} P_n(0)P_n(x) = \delta(x). \]

Concerning the weighted Laguerre polynomials, \( P_n(x) = e^{-x/2}L_n(x) \), which are defined on \([0, \infty)\) with \( P_n(0) = 1 \), one notes that the infinite sum expansion
\[ \sum_{n=0}^{\infty} L_n(x) = \delta_+(x) \quad (4) \]
is a representation of a GF (we will come back to this point in the next section, with a proper demonstration)\(^3\). The GF on the right-hand side (RHS) of equation (4) is related to the Dirac distribution
\[ \int_0^\infty \delta_+(x)\phi(x) \, dx = \lim_{\epsilon \to 0^+} \int_0^\infty \delta(x - \epsilon)\phi(x) \, dx = \phi(0) \quad (5) \]
since GFs are properly defined in open intervals.

The infinite sums of polynomials and moments are useful for a certain class of problems, as we will see in section 4. In what follows, we shall consider the associated Laguerre polynomials \( L_\alpha^n(x) \), whose generating function (GEF) is
\[ G(x, t, \alpha) = e^{-xt}(1 - t)^{\alpha+1} \quad (6) \]
since
\[ G(x, t, \alpha) = \sum_{n=0}^{\infty} L_\alpha^n(x)t^n \quad |t| < 1 \quad (7) \]
where \( t \) is a complex variable in the open disc of radius \(|t|, t \in D_t = \{ t \in \mathbb{C} \mid 0 \leq |t| < 1 \} \).

In section 2 we present some lemmas involving the ordinary Laguerre polynomials and the theorem for the Laguerre moments, whereas in section 3 we extend the results to the associated Laguerre polynomials. In section 4 we make use of previous results and obtain an expression for the P-distribution in terms of either the Laguerre factorial moments or the GFs. In section 5 we expose our conclusions.

2. The ordinary Laguerre polynomials and moments

Initially, we shall consider the ordinary Laguerre polynomials \( (\alpha = 0) \), \( L_0^n(x) \equiv L_n(x) \), where \( L_n(0) = 1 \). If we extend the domain of \( t \) to include the additional point \( t = +1 \) in

\(^3\) Here the \( \delta_+(x) \) should not be confused with the distributions \( \delta^{\pm} = \tfrac{1}{2\pi i} \pm \frac{1}{2\pi i} P \) defined in [6], p 91, where \( P \) stands for the Cauchy principal value.
equations (6) and (7) then $\mathcal{D}_t = \{D_t, 1\}$; one verifies that at this point (since $t = |t| e^{i\varphi}$, $|t| = 1$ and $\varphi = 0$) the GEF (6) becomes

$$
\lim_{t \to 1^-} G(x, t, 0) = \lim_{t \to 0^+} \frac{e^{-\frac{x}{t}}}{t} = \begin{cases} 0 & \text{for } x > 0 \\ \infty & \text{for } x = 0 \end{cases}.
$$

(8)

Thus $G(x, 1, 0)$ is no longer a regular function in the usual sense, it becomes quite singular at $x = 0$. Let us look more closely at GEF (6) and analyse its properties:

**Lemma 1.** For $x \in \mathbb{R}_+$, $\mathbb{R}_+ \equiv (0, \infty)$, the $\alpha = 0$ GEF $G(x, 1, 0)$ can be represented by the GF, equation (5),

$$
G(x, 1, 0) = \delta_+(x).
$$

(9)

**Proof.** Multiplying $G(x, t, 0)$ by a piecewise smooth test function $\phi(x)$, with $\phi \in \mathbb{R}$ and $x \in \mathbb{R}_+$, integrating

$$
\int_0^\infty e^{-\frac{y}{t}} \phi(x) \frac{1}{(1-t)} \, dx
$$

performing the change of variable $x = y(1-t)/t$ and considering the limit $t \to 1^-$, we get

$$
\lim_{t \to 1^-} \int_0^\infty e^{-\frac{1}{t}} \phi \left( \frac{1-t}{t} y \right) \frac{1}{(1-t)} \, dy = \phi(0)
$$

which is the main property of the GF, thus

$$
\lim_{t \to 0^+} \int_0^\infty G(x, 1 - \varepsilon, 0) \phi(x) \, dx = \phi(0)
$$

(10)

and equation (9) is justified, i.e., $\lim_{t \to 1^-} G(x, t, 0)$ is a representation of the GF. □

Symbolically, equation (10) can be written as

$$
G(x, 1, 0) \phi(x) = \phi(0) \quad \text{or} \quad \delta_+(x) \phi(x) = \phi(0)
$$

which is a well-known property of the DD, omnipresent in mathematical physics textbooks [6, 9, 10].

As the summation $\sum_{n=0}^N L_n(x)$ is a regular function for any finite $N$, we may ask: does the infinite summation go to a GF? Or, is the equality $\sum_{n=0}^\infty L_n(x) = \delta_+(x)$ true? Before answering that question we first recall the following theorem of the Laguerre polynomials (we do not present the demonstration since it can be found in the usual textbooks [11]):

**Theorem 1.** If the real function $f(x)$, defined in the interval $[0, \infty)$, is piecewise smooth in every subinterval $[x_1, x_2]$, where $0 \leq x_1 < x_2 < \infty$ and if the integral

$$
\int_0^\infty e^{-x} x^\alpha \left| f(x) \right|^2 \, dx
$$

is finite, then the series

$$
\sum_{n=0}^\infty c_{n, \alpha} L_n^\alpha(x)
$$

with coefficients

$$
c_{n, \alpha} = \frac{n!}{\Gamma(n + \alpha + 1)} \int_0^\infty e^{-x} x^\alpha f(x) L_n^\alpha(x) \, dx
$$
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converges to \( f(x) \) at every continuity point of \( f(x) \). At a discontinuity point \( x_0 \) the series converges to

\[
\frac{1}{2} \lim_{\varepsilon \to 0} [f(x_0 + \varepsilon) + f(x_0 - \varepsilon)].
\]

For \( \alpha = 0 \) we have

\[
f(x) = \sum_{n=0}^{\infty} c_n L_n(x) \tag{11}
\]

with

\[
c_n = \int_0^{\infty} e^{-x} f(x) L_n(x) \, dx. \tag{12}
\]

The Laguerre polynomials are defined such that \( L_n(0) = 1 \), implying that \( f(0) = \sum_{n=0}^{\infty} c_n \); so we propose

**Lemma 2.** From equation (12) and for \( x \in \mathbb{R}_+ \equiv (0, \infty) \) we obtain

\[
\sum_{n=0}^{\infty} L_n(x) = \delta_+(x). \tag{13}
\]

**Proof.** Summing over all \( n \) on both sides of equation (12) and interchanging the order of summation and integration, we get

\[
\sum_{n=0}^{\infty} c_n = f(0) = \int_0^{\infty} e^{-x} f(x) \left( \sum_{n=0}^{\infty} L_n(x) \right) \, dx
\]

so we verify equation (13). \( \square \)

We can also verify equation (13) by using a recurrence relation of the Laguerre polynomials and a property of the DD:

**Corollary 1.** From the recurrence relation of the Laguerre polynomials

\[
xd_x L_n(x) = nL_n(x) - nL_{n-1}(x) \quad n = 1, 2, 3, \ldots \tag{14}
\]

where \( d_x \equiv \frac{d}{dx} \), follows the known relation of the DD

\[
xd_x \delta_+(x) = -\delta_+(x) \tag{15}
\]

for \( \sum_{n=0}^{\infty} L_n(x) = \delta_+(x) \).

**Proof.** Summing both sides of equation (14) over \( n \), from 1 to \( N \), we get

\[
\sum_{n=1}^{N} nL_n(x) = xd_x \sum_{n=1}^{N} L_n(x) + \sum_{n=1}^{N} nL_{n-1}(x)
\]

or

\[
\sum_{n=0}^{N} nL_n(x) = xd_x \sum_{n=0}^{N} L_n(x) + \sum_{n=0}^{N} (n + 1)L_n(x)
\]

then

\[
xd_x \left( \sum_{n=0}^{N} L_n(x) \right) = -\sum_{n=0}^{N} L_n(x).
\]

Considering \( N \to \infty \) we recognize equation (15) for \( \sum_{n=0}^{\infty} L_n(x) = \delta_+(x) \). \( \square \)
Therefore, we can write
\[ G(x, 1, 0) = \sum_{n=0}^{\infty} L_n(x) = \delta_+(x). \] (16)

We now give some relations involving the GFs that will be necessary to demonstrate a useful theorem. As a preliminary, we

(i) write in short \( \delta(n)^+ + (x) \).

(ii) assume \( \lim_{x \to 0} (x \sum_{n=0}^{\infty} L_n(x)) = 0 \) and \( \lim_{x \to 0} \left( x (d_1)^n \sum_{m=0}^{\infty} L_m(x) \right) = 0 \), thus

\[ \lim_{x \to 0} x \delta(n)^+(x) = \lim_{x \to 0} x (d_1)^n G(x, 1, 0) = 0; \] (17)

(iii) introduce the bracketed terms \( [x \delta(n)^+(x)] \), \( [x \delta(n)^2(x)] \), \ldots, \( [x \delta(n)^k(x)] \) as GFs;

(iv) define the functional

\[ \left( [x \delta(n)^+(x)], \phi \right) = \int_{0}^{\infty} \left[ x \delta(n)^+(x) \right] \phi(x) \, dx \]

where \( \phi(x) \) is a regular piecewise and bounded test function in \( \mathbb{R}_+ \).

**Example 1.** For \( n = 1 \), the functional is

\[ \int_{0}^{\infty} \left[ x \delta(1)^+(x) \right] \phi(x) \, dx = \int_{0}^{\infty} \delta(1)^+(x) \phi(x) \, dx = - \int_{0}^{\infty} \delta_+(x) \left[ d_1 (\phi(x)x) \right] \, dx \]

or in symbolic notation

\[ [x \delta(1)^+(x)] = - \delta_+(x) \] (18)

the term in the brackets is reduced to the GF multiplied by \(-1\).

**Example 2.** For \( n = 2 \),

\[ \int_{0}^{\infty} \left[ x \delta(2)^+(x) \right] \phi(x) \, dx = \int_{0}^{\infty} \delta(2)^+(x) \phi(x) \, dx = (-1)^2 \int_{0}^{\infty} \delta_+(x) \left[ (d_1)^2 (\phi(x)x) \right] \, dx \]

\[ = \int_{0}^{\infty} \delta_+(x) \left[ x (d_1)^2 \phi(x) + 2 d_1 \phi(x) \right] \, dx = 2 \int_{0}^{\infty} \delta_+(x) \left[ d_1 \phi(x) \right] \, dx \]

\[ = \int_{0}^{\infty} (-2 \delta(1)^+(x)) \phi(x) \, dx = 2 \phi'(0) \]

therefore, in symbolic notation

\[ [x \delta(2)^+(x)] = - 2 \delta(1)^+(x) \] (19)

**Remark 1.** The bracket \( [x \delta_+(x)] = 0 \), since

\[ \int_{0}^{\infty} [x \delta_+(x)] \phi(x) \, dx = \int_{0}^{\infty} \delta_+(x) \phi(x) \, dx = 0. \]

The general term \([x \delta(n)^+(x)]\) is obtained by induction:

**Lemma 3.** For \( x \in \mathbb{R}_+ \), the factor \((-x)\) in \([(-x) \delta(n)^+(x)]\) acts as a first-order derivative on \( \delta(n)^+(x) \),

\[ [x \delta(n)^+(x)] = - n \delta(n-1)^+(x) \quad n = 1, 2, 3, \ldots. \] (20)
Proof. From examples 1 and 2 we have \( [x \delta^{(1)}_+(x)] = -\delta_+(x) \) and \( [x \delta^{(2)}_+(x)] = -2\delta^{(1)}_+(x) \). For \( [x \delta^{(3)}_+(x)] \) we use this last equation,
\[
[x \delta^{(3)}_+(x)] = d_1 \left[ x \delta^{(2)}_+(x) \right] - \delta^{(1)}_+(x) = d_1 ( -2\delta^{(1)}_+(x) ) - \delta^{(2)}_+(x) = -3\delta^{(2)}_+(x)
\]
and so forth for higher order derivatives, so equation (20) stands for any positive integer \( n \).
\[ \square \]

We can generalize this result for higher powers of \( x \) through

**Lemma 4.** For \( x \in \mathbb{R}_+ \), any positive integer \( p \), and assuming relation (20), it follows that in \( [(-x)^p \delta^{(n)}_+(x)] \) the factor \((-x)^p\) acts as a \( p \)-th order derivative multiplied by a constant,
\[
[x^p \delta^{(n)}_+(x)] = \begin{cases} (-1)^p \frac{n!}{(n-p)!} \delta^{(n-p)}_+(x) & \text{for } n \geq p \\ 0 & \text{for } n < p. \end{cases}
\]  

Proof. Since \( [x \delta^{(n)}_+(x)] = -n\delta^{(n-1)}_+(x) \) then
\[
x^p \delta^{(n)}_+(x) = [x(-n \delta^{(n-1)}_+(x))] = -n[x \delta^{(n-1)}_+(x)] = -n(-n+1)\delta^{(n-2)}_+(x) \quad \text{for } n \geq 2.
\]
However, \( [x^2 \delta^{(1)}_+(x)] = -[x \delta_+(x)] = 0 \), where the second equality follows from remark 1. Repeating this procedure for any positive integer \( p \), we verify equation (22).
\[ \square \]

**Lemma 5.** For the differential operator
\[
\Lambda(x) \doteq -((1-x)d_x + xd_x^2)
\]
the following equation
\[
[\Lambda(x) \delta^{(n)}_+(x)] = (n+1) \left( \delta^{(n+1)}_+(x) - \delta^{(n)}_+(x) \right)
\]  
holds for \( n = 0, 1, 2, \ldots \)

Proof. Setting \( n = 0 \) in equation (23) and by using relation (22) we get
\[
[\Lambda(x) \delta_+(x)] = -\left[ \left(1-x\right) \delta^{(1)}_+(x) \right] + \left[ x \delta^{(2)}_+(x) \right] = -\delta^{(1)}_+(x) - (-\delta_+(x)) + \left(\frac{-2}{1!}\delta^{(1)}_+(x)\right) = \delta^{(1)}_+(x) - \delta_+(x).
\]  
\[ \square \]

Following the same procedure for \( n \geq 1 \) we get equation (23).

**Lemma 6.** For any positive integer \( s \) we have
\[
[(\Lambda(x))^s \delta^{(l)}_+(x)] = \prod_{k=1}^{x} \left( r + k \right) \delta^{(s+r)}_+(x) - \prod_{k=1}^{x-1} \left( r + k \right) \sum_{j=1}^{x-1} \left( r + j \right) \delta^{(s+r-1)}_+(x)
\]  
\[ + \prod_{k=1}^{x-2} \left( r + k \right) \sum_{j=1}^{x-2} \left( r + j \right) \sum_{l=1}^{x-2} \left( r + l \right) \delta^{(s+r-2)}_+(x) - \prod_{k=1}^{x-3} \left( r + k \right) \sum_{j=1}^{x-3} \left( r + j \right) \sum_{l=1}^{x-3} \left( r + l \right) \sum_{m=1}^{x-3} \left( r + m \right) \delta^{(s+r-3)}_+(x) + \cdots
\]  
\[ + \left( -1 \right)^s \prod_{j=1}^{x} \left( r + j \right) \sum_{l=1}^{x} \sum_{m=1}^{x} \cdots \sum_{n=m}^{x} \frac{1}{n!} \delta^{(r)}_+(x). \]  

(25)
**Proof.** Starting with equation (23) and applying $\Lambda(x)$ successively, by induction we arrive at equation (25).

We now define the Laguerre moments

$$\mathcal{M}_s(x, 0) \doteq \sum_{n=0}^{\infty} n^s L_n(x)$$

and propose the following theorem.

**Theorem 2.** For any positive integer $s$ and $x \in \mathbb{R}_+$, the moments $\mathcal{M}_s(x, 0)$ can be expanded in terms of GFs as

$$\mathcal{M}_s(x, 0) = \sum_{l=0}^{s} c_{s,l} \delta_x^{(l)}(x)$$

with the coefficients given by

$$c_{s,l} = (-1)^{s-l} l! \sum_{j=1}^{l+1} \sum_{k=j}^{l+1} \sum_{m=k}^{l+1} \sum_{n=m}^{l+1} n.$$  

**Proof.** The eigenvalue equation for the Laguerre polynomials is

$$\Lambda(x)L_n(x) = nL_n(x).$$

Summing both sides of the equation over $n$ from 0 to $\infty$ gives the first moment,

$$\mathcal{M}_1(x, 0) = \sum_{n=0}^{\infty} nL_n(x) = [\Lambda(x)] = \delta_x^{(1)}(x) - \delta_x(x)$$

where we used equation (24) to obtain the second equality. By applying $s-1$ times $\Lambda(x)$ on both sides of equation (29) we obtain

$$(\Lambda(x))^s L_n(x) = n^s L_n(x)$$

summing over $n$ we get the $s$th-order moment

$$\mathcal{M}_s(x, 0) = \sum_{n=0}^{\infty} n^s L_n(x) = [(\Lambda(x))^s \delta_x(x)].$$

Setting $r = 0$ in equation (25) we get

$$\mathcal{M}_s(x, 0) = s! \delta_x^{(s)}(x) - (s-1)! \left( \sum_{j=1}^{s} \sum_{j=1}^{s} \sum_{k=j}^{s} \sum_{m=k}^{s} \sum_{n=m}^{s} n \right) \delta_x^{(s-1)}(x) + (s-2)! \left( \sum_{j=1}^{s-2} \sum_{j=1}^{s-2} \sum_{k=j}^{s-2} \sum_{m=k}^{s-2} \sum_{n=m}^{s-2} n \right) \delta_x^{(s-2)}(x)$$

$$- (s-3)! \left( \sum_{j=1}^{s-3} \sum_{j=1}^{s-3} \sum_{k=j}^{s-3} \sum_{m=k}^{s-3} \sum_{n=m}^{s-3} n \right) \delta_x^{(s-3)}(x) + \cdots + (-)^s \delta_x(x)$$

$$= \sum_{l=0}^{s} \left[ (-1)^{s-l} l! \sum_{j=1}^{l+1} \sum_{j=1}^{l+1} \sum_{k=j}^{l+1} \sum_{m=k}^{l+1} \sum_{n=m}^{l+1} n \right] \delta_x^{(l)}(x) = \sum_{l=0}^{s} c_{s,l} \delta_x^{(l)}(x).$$

□
Example 3.
\[ M_2(x, 0) = \left[ \Lambda(x) \delta_s(x) \right] = \left[ \Lambda(x) \left( \delta_s(x) - \delta_s(x) \right) \right] = 2 \left( \delta_s(x) - \delta_s(x) \right) + \delta_s(x) \]
\[ = 2\delta_s(x) - 3\delta_s(x) + \delta_s(x) \]
proceeding analogously we get
\[ M_3(x, 0) = 6\delta_s(x) - 12\delta_s(x) + 7\delta_s(x) - \delta_s(x) \]
and
\[ M_4(x, 0) = 24\delta_s(x) - 60\delta_s(x) + 50\delta_s(x) - 15\delta_s(x) + \delta_s(x). \]
It can be verified that
\[ \sum_{l=0}^{s} c_{s,l} = 0. \]

2.1. The Laguerre factorial moments

The Laguerre factorial moment of order \( s \), associated with the ordinary Laguerre polynomials, is defined as
\[ F_s(x, 0) = \sum_{n=0}^{\infty} n(n-1)(n-2) \cdots (n-s+1) L_n(x) = \sum_{n=s}^{\infty} \frac{n!}{(n-s)!} L_n(x) \]
and it is related to the Laguerre moments through
\[ F_s(x, 0) = \sum_{r=1}^{s} S^{(s)}(r) M_r(x, 0) \]
where the coefficients \( S^{(s)}(r) \) are the Stirling numbers of the first kind \[12]^4. By substituting equation (27) into (35) and after summing over the coefficients we arrive at a simple expression for the factorial moments in terms of ultradistributions,
\[ F_s(x, 0) = s! \sum_{r=0}^{s} (-1)^{s-r} \binom{s}{r} \delta_s(x). \]

3. The associated Laguerre polynomial and moments

Lemma 7. For \( \alpha \) a positive integer and \( x \in \mathbb{R}_+ \), the equation
\[ \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) = (1 - d_x)^\alpha \delta_s(x) \]
holds and for \( t = 1 \) the GEF also becomes
\[ G(x, 1, \alpha) = \lim_{t \to 1} \frac{e^{-\frac{xt}{1-t}}}{{\alpha + 1}} = (1 - d_x)^\alpha \delta_s(x). \]

\[ ^4 \text{The Stirling numbers of the first kind have the properties } S^{(0)}(s, 0) = \delta_s \text{ and } \sum_{r=0}^{s} S^{(r)}(s) = 0. \text{ The inverse relation is } M_r(x, 0) = \sum_{s=0}^{r} S^{(s)}(r) F_s(x, 0) \text{ where } S^{(s)}(r) \text{ are the Stirling numbers of the second kind.} \]
Proof. Considering the following properties of the Laguerre polynomials, [13, 14]
\[ L_α^n(x) = (-d)_x^α L_{n+α}(x) \quad \text{and} \quad (d)_x^α L_{n+α}(x) = (-1 + d)_x^α L_n(x) \]
it follows that
\[ L_α^n(x) = (1 - d)_x^α L_n(x) \quad (40) \]
and summing both sides of equation (40) over \( n \) from 0 to \( ∞ \) one gets equation (38).

Concerning equation (39), looking at \( G(x, t, α) \), equation (7), one can easily verify that
\[ G(x, t, α) = (1 - ∂_x)^α G(x, t, 0) \quad (41) \]
\( (\partial_x \equiv \partial/∂x) \) thus
\[ \lim_{t \to 1} G(x, t, α) = (1 - d)_x^α \lim_{t \to 1} G(x, t, 0) = (1 - d)_x^α δ_α(x) \]
(the second equality follows from lemma 1) so verifying equation (39). Therefore, one arrives at
\[ G(x, 1, α) = \sum_{n=0}^{∞} L_α^n(x) = (1 - d)_x^α δ_α(x) \quad (42) \]
which generalizes equation (9). □

We now propose the following theorem.

Theorem 3. For any positive integers \( α \) and \( s \), and \( x \in \mathbb{R}_+ \), the moments \( M_s(x; α) \) are given by the following summation of GFs:
\[ M_s(x; α) = \sum_{n=0}^{∞} n^s L_α^n(x) = \sum_{l=0}^{s} c_{s,l} \sum_{j=0}^{α} (-1)^j \left( α \atop j \right) δ_α^{(l+j)}(x). \quad (43) \]

Proof. Considering equation (40), multiplying both sides by \( n^s \) (\( s \) a non-negative integer) and summing over \( n \) from 0 to \( ∞ \), one gets
\[ M_s(x; α) = (1 - d)_x^α M_s(x; 0) = (1 - d)_x^α \sum_{l=0}^{s} c_{s,l} δ_α^{(l)}(x) \]
\[ = \sum_{l=0}^{s} c_{s,l} \sum_{j=0}^{α} (-1)^j \left( α \atop j \right) δ_α^{(l+j)}(x). \quad (44) \]
□

3.1. The factorial moments

The associated Laguerre factorial moment of order \( s \) is defined as
\[ \mathcal{F}_s(x, α) = \sum_{n=0}^{∞} \frac{n^s}{(n-s)!} L_α^n(x) \quad (45) \]
and using relations (44) we get
\[ \mathcal{F}_s(x, α) = (1 - d)_x^α \mathcal{F}_s(x, 0) \]
\[ = s! \sum_{r=0}^{s} (-1)^{r-s} \binom{s}{r} \sum_{j=0}^{α} (-1)^j \left( α \atop j \right) δ_α^{(r+j)}(x). \quad (47) \]
4. The P-distribution for the photon-number state

Writing the complex variable $\alpha$ in the polar form and calling $|\alpha|^2 = y$ in equation (2) we obtain

$$Q_n(y) = \frac{\exp(-y)y^n}{n!}$$

and

$$P_n(y) = \frac{1}{n!} \exp\left(-\frac{y}{\beta}\frac{d}{dy}\right) y^n \exp(-y).$$  \hspace{1cm} (48)

It is convenient to introduce an auxiliary function,

$$R(y, \beta) = \exp\left(-\frac{y}{\beta}\frac{d}{dy}\right) \exp(-y\beta)$$  \hspace{1cm} (49)

and since

$$\left(-\frac{d}{dy}\frac{d}{dy}\right)^n \exp(y) = (-1)^n n! L_n(y) \exp(y)$$

we can write

$$R(y, \beta) = \sum_{n=0}^{\infty} \beta^n L_n(\beta y) \exp(-y\beta).$$  \hspace{1cm} (50)

Now we can express equation (48) in terms of equation (50) as

$$P_n(y) = \frac{1}{n!} \lim_{\beta \to 1} \left(-\frac{d}{\partial \beta}\right)^n R(y, \beta)$$

$$= \frac{1}{n!} \lim_{\beta \to 1} \left(-\frac{d}{\partial \beta}\right)^n \left[\exp(-y\beta) \sum_{k=0}^{\infty} \beta^k L_k(\beta y)\right]$$  \hspace{1cm} (51)

thus the auxiliary function $R(y, \beta)$ stands for the GEF of the P-distributions. Using the definition of GEF (7) we have

$$\sum_{k=0}^{\infty} \beta^k L_k(\beta y) = G(y, \beta, 0) = \sum_{k=0}^{\infty} \beta^k L_k(y)$$  \hspace{1cm} (52)

and substituting this result in equation (51) we obtain

$$P_n(y) = (-1)^n \sum_{k=n}^{\infty} \left(\frac{k}{n}\right) L_k(y).$$  \hspace{1cm} (53)

Using equations (35) and (37) we get a direct relation between the P-distributions and the Laguerre factorial moments,

$$P_n(y) = \frac{(-1)^n}{n!} \mathcal{F}_n(y, 0)$$  \hspace{1cm} (54)

or in terms of the GFs

$$P_n(y) = \sum_{k=0}^{n} \left(-1\right)^k \binom{n}{k} \delta_k(y) = (1 - d_y)^n \delta_n(y)$$  \hspace{1cm} (55)

which simplifies the task of calculating mean values in equation (1) when the state of the field can be written as $\hat{\rho} = \sum_{n=0}^{\infty} p_n |n\rangle \langle n|$, where $p_n$ is the probability associated with the state $|n\rangle$. As an illustration, we display the $n = 0, 1, 2$ P-distributions,

$$P_0(y) = \delta_n(y) \quad P_1(y) = -\delta_n^{(1)}(y) + \delta_n(y) \quad P_2(y) = \delta_n^{(2)}(y) - 2\delta_n^{(1)}(y) + \delta_n(y).$$
5. Conclusions

We have obtained an explicit expression for the Laguerre moments and factorial moments, for ordinary and associated Laguerre polynomials, written as sums of generalized functions. We showed that the P-distribution of an electromagnetic field in a photon-number state $|n\rangle$ is proportional to the Laguerre factorial moment of order $n$ and that it acquires a simple form when expressed as a sum of generalized functions.

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