INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 427–448 doi:10.1088/0305-4470/38/2/010

O(N) symmetries, sum rules for generalized Hermite polynomials and squeezed states

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Received 12 July 2004, in final form 4 November 2004
Published 15 December 2004
Online at stacks.iop.org/JPhysA/38/427

Abstract

Quantum optics has been dealing with coherent states, squeezed states and many other non-classical states. The associated mathematical framework makes use of special functions as Hermite polynomials, Laguerre polynomials and others. In this connection we here present some formal results that follow directly from the group $O(N)$ of complex transformations. Motivated by the squeezed states structure, we introduce the *generalized Hermite polynomials* ($\mathcal{GHP}$), which include as particular cases, the Hermite polynomials as well as the heat polynomials. Using generalized raising operators, we derive new sum rules for the $\mathcal{GHP}$, which are covariant under $O(N)$ transformations. The $\mathcal{GHP}$ and the associated sum rules become useful for evaluating Wigner functions in a straightforward manner. As a byproduct, we use one of these sum rules, on the operator level, to obtain raising and lowering operators for the Laguerre polynomials and show that they generate an $sl(2, R) \simeq su(1, 1)$ algebra.

PACS numbers: 02.20.Sv, 02.10.Nj, 42.50.Dv

1. Introduction

Quantum optics has been dealing with coherent states, squeezed states (of the harmonic oscillator—HO) and many other non-classical states [1]. The associated mathematical framework makes use of special functions as Hermite polynomials, Laguerre polynomials and others, which have been playing an important role in the field [2–6]. In particular, the coherent states are generated by exponentials of boson rising operators of the Weyl–Wigner algebra, and their wavefunctions, in the x-representation, depend on Hermite polynomials. These polynomials are also encountered when dealing with Wigner functions [3]. Therefore, when working with coherent states or with Wigner functions, one frequently encounters complicated relationships involving Hermite, and also Laguerre, polynomials. Such relations will be called *sum rules* or *addition theorems*.

For example, in order to derive the wavefunction of nth HO state $|n\rangle$ in the squeezed-state representation $\langle p, q; \lambda|n\rangle$ (see section 5.2 below), Schleich, Walls and Wheeler [7] used the following sum rule:

$$(-4)^n L_n(x^2+y^2) = \sum_{k=0}^{n} \frac{1}{(n-k)!k!} H_{2n-2k}(x) H_{2k}(y), \tag{1}$$

where $L_n(x)$ and $H_n(x)$ are the Laguerre and the Hermite polynomials. This formula follows immediately by substituting the well-known equality [8, section 8.972]

$$H_{2n}(x) = (-4)^n n! L_n^{-\frac{1}{2}}(x^2) \tag{2}$$

into the following addition formula for the *associated Laguerre polynomials* $L_n^{\alpha}(x)$ [8, section 8.974]:

$$L_n^{\alpha+\beta+1}(x+y) = \sum_{k=0}^{n} L_{n-k}^{\alpha}(x) L_k^{\beta}(y). \tag{3}$$

While working on squeezed states, one of us [9] was led to conjecture the validity of the sum rule

$$\frac{1}{n!2^n} H_n(\sqrt{2}x) H_n(\sqrt{2}y) = \sum_{k=1}^{n} (-1)^k L_{n-k}^{-\frac{1}{2}}((x+y)^2) L_k^{-\frac{1}{2}}((x-y)^2). \tag{4}$$

By substituting relation (2) into equation (4), we arrive at the following interesting sum rule for Hermite polynomials:

$$\frac{2^n}{n!} H_n\left(\frac{x+y}{\sqrt{2}}\right) H_n\left(\frac{x-y}{\sqrt{2}}\right) = \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!k!} H_{2n-2k}(x) H_{2k}(y). \tag{5}$$

We shall call equation (5) the *factorization sum rule*. Note that the rhs of equations (1) and (5) differ only by the sign factor $(-1)^k$. However, their similarity is deceptive. In fact, it turns out that these two sum rules have completely different origins, as we shall see below. Equation (5) can be checked explicitly for $n = 1, 2, 3$, but such verification becomes tedious, already for $n = 4$.

We here prove equation (5) by first recognizing that the raising operators for the Hermite polynomials transform covariantly under the group of complex rotations $O(N)$, and then using the binomial expansion (15) of powers of sums of raising operators. Subsequently, we generalized this technique to N variables, and thus developed a powerful general procedure for deriving new and more complicated sum rules.

In the present paper we describe the above technique and explain the symmetry hidden behind the sum rules. As illustrations, we derive several new sum rules, and give new proofs to already known sum rules, which are usually proved by using the generating function method. As a natural extension of those techniques, we define *generalized Hermite polynomials* ($\mathcal{GHP}$), which include, as special cases: the standard Hermite polynomials, the heat polynomials [10, 11], as well as simple powers.

By utilizing one of such sum rule, at the operator level, we derive rising and lowering operators for the Laguerre polynomials, and show that they generate an $sl(2, R) \simeq su(1, 1)$ algebra [12]. The Laguerre polynomials are also shown to yield an infinite-dimensional irreducible representation of $sl(2, R)$ group.

The paper is organized as follows: in section 2 we define the generalized raising operators that depend on the usual variable x, plus two (complex) parameters, α and β. We then show the covariance of these operators under complex rotations. By making use of the raising-operator techniques, in section 3 we prove old and new sum rules which show to be useful to derive several secondary sum rules. In section 4, we derive raising and lowering operators for the Laguerre polynomials, and show that they generate an $sl(2, R)$ algebra. In section 5 we present some applications: We first use the factorization sum rule (5) to evaluate an integral, which in turn is used to obtain the Wigner function of the HO wavefunctions. Then we demonstrate three unfamiliar applications of the $\mathcal{GHP}$, one of which involves squeezed states. Finally, in section 6 we give a summary of our work. In appendix A we give various properties of the $\mathcal{GHP}$: we define the $\mathcal{GHP}$ via the raising operators, derive scaling and symmetry properties; we obtain a lowering operator for the $\mathcal{GHP}$; we derive the generating function, and point out two methods for obtaining the power expansion for the $\mathcal{GHP}$. In appendix B we relate special $\mathcal{GHP}$ to some standard known polynomials.

2. Raising operators and their covariance transformation under complex orthogonal rotations

Although the *real* orthogonal groups are, by far, quite familiar, we shall show that most of the known results and sum rules are automatically valid also for the less familiar complex orthogonal group $O(N) \equiv O(N, \mathbb{C})$. Therefore, we shall be working with complex rotations $O(N)$ throughout the present paper, unless stated otherwise.

The *group of complex rotations* $O(N)$ is defined as the group of all the $N \times N$ *complex* matrices O, such that $OO^T = O^T O = 1$, where T means transposed. These transformations preserve the real scalar product, $x \cdot y := \sum_{i=1}^{N} x_i y_i$, so that $(Ox) \cdot Oy = x \cdot y$, where x and y are complex vectors in general, i.e., the components $x_i, y_i \in \mathbb{C}$. Note that, in contrast to the real orthogonal groups, the complex $O(N)$ groups are not subgroups of the corresponding unitary groups $SU(N)$.

For example, the following linear transformation:

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \cosh\chi & \mathrm{i}\sinh\chi \\ -\mathrm{i}\sinh\chi & \cosh\chi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{6}$$

is a complex orthogonal rotation, which is not unitary. In fact, if we make the identification $x := (x_1, x_2) \equiv (z, \mathrm{i}ct)$, (6) describes a Lorentz boost in two space–time dimensions.

Under complex orthogonal coordinate transformations $O \in O(N)$:

$$w_i := (Ox)_i = \sum_{j=1}^{N} O_{ij} x_j, \qquad \text{for} \quad i = 1, \ldots, N, \tag{7}$$

the partial derivatives

$$\frac{\partial}{\partial w_i} = \sum_{j=1}^{N} \frac{\partial x_j}{\partial w_i} \frac{\partial}{\partial x_j} = \sum_{j=1}^{N} O^{-1}_{ji} \frac{\partial}{\partial x_j} = \sum_{j=1}^{N} O_{ij} \frac{\partial}{\partial x_j}, \tag{8}$$

transform exactly as the coordinates, because of the property $O^{-1} = O^T$ of orthogonal matrices. This is the mathematical reason why in quantum mechanics the components of the linear momentum operator, $p_i = -\mathrm{i}\hbar \frac{\partial}{\partial x_i}$, in coordinate representation, transform like the coordinates under rotations. However, the p_i do not transform like the coordinates x_i under unitary transformations.

Therefore, any *linear* combination of the coordinates and the partial derivatives,

$$R(\alpha, \beta; x_i) := \alpha x_i - \beta \frac{\partial}{\partial x_i}, \tag{9}$$

must transform like the coordinates, for any complex parameters α and β:

$$R(\alpha, \beta; w_i) := \alpha w_i - \beta \frac{\partial}{\partial w_i} = \sum_{j=1}^{N} O_{ij} \left(\alpha x_j - \beta \frac{\partial}{\partial x_j} \right) = \sum_{j=1}^{N} O_{ij} R(\alpha, \beta; x_j). \tag{10}$$

We call such linear combinations (9) *raising operators*. They generate infinite sequences of special functions $g_n(x) = R^n(x) g_0(x)$, when applied to appropriate functions $g_0(x)$, which we call *vacuum* or *ground-state* functions.

2.1. Illustrative examples

Before explaining the above formalism further, it is useful to keep the following three examples in mind:

(1) *The standard Hermite polynomials.* These polynomials satisfy the well-known recursion formula $H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x)$ and the relation $\frac{\partial H_n(x)}{\partial x} = 2n H_{n-1}(x)$. Combining these two relations, we get

$$H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x) = \left(2x - \frac{\partial}{\partial x} \right) H_n(x) = R(2, 1; x) H_n(x) \tag{11}$$

which shows that the usual Hermite polynomials can be generated by applying the raising operator $R(2, 1; x) := 2x - \frac{\partial}{\partial x}$ to the ground-state function $g_0(x) = H_0(x) = 1$.

(2) *The eigenfunctions of the HO.* The raising operators for this system are the usual *creation operators* quite familiar from quantum mechanics, $a^\dagger(x) := \left(x - \frac{\partial}{\partial x} \right) / \sqrt{2} = R(1/\sqrt{2}, 1/\sqrt{2}; x)$. Here, the ground-state function is $g_0(x) = \mathrm{e}^{-x^2/2}$ is that of the HO. The excited states are given by $\psi_n(x) \propto g_n(x) = (a^\dagger)^n \, \mathrm{e}^{-x^2/2} = 2^{-n/2} H_n(x) \, \mathrm{e}^{-x^2/2}$.

(3) *Powers of x.* For $\beta = 0$ and $g_0 = 1$, we get $R(\alpha, 0; x) = \alpha x$, so that $g_n(x) = \alpha^n x^n$.

2.2. Standard versus generalized Hermite polynomials

We now use the raising operator (9) to define *generalized Hermite polynomials* as

$$\tilde{H}_n(\alpha, \beta; x) := R^n(\alpha, \beta; x) \cdot 1. \tag{12}$$

The $\mathcal{GHP}$ polynomials are derived in appendix A, and their explicit power expansion is given in equation (A.3). Besides the standard Hermite polynomials, these $\mathcal{GHP}$ include the heat polynomials (see equation (A.3) and section 5.2 below) and simple powers of x, as particular cases. Their properties and applications are presented and discussed in the appendices.

Because of the scaling relation,

$$R(\alpha, \beta; x_i) = \sqrt{\alpha\beta} R(2, 1; \tilde{x}_i) \qquad \text{where} \qquad \tilde{x}_i := \sqrt{\frac{\alpha}{\beta}} x_i, \qquad \text{for} \quad \alpha, \beta \neq 0, \tag{13}$$

the generalized raising operator (9) is proportional to a standard raising operator, say $R(2, 1; \tilde{x}_i)$, but with a re-scaled variable $\tilde{x}_i$. In this way, we can relate the $\mathcal{GHP}$ to the standard Hermite polynomials, as in equation (A.11). However, by keeping β as a free parameter we have two main advantages:

(1) It allows us to take the zero limit, $\beta \to 0$, without getting involved with ratios of infinities, since the re-scaled variable $\tilde{x}_i$ blows up in this limit: $\tilde{x}_i = \sqrt{\alpha/\beta} x_i \to \infty$.

(2) For real and negative values of β and $\alpha > 0$, which is needed to get the heat equation, the re-scaled variable $\tilde{x}_i$ becomes pure imaginary. So, by keeping β as a free parameter, we avoid using complex arguments.

In contrast, the freedom of choosing $\alpha \neq 0$ is not important, because α serves essentially as a normalization factor. Therefore, we could have chosen a fixed α throughout this paper to shortcut the notation. But we decided to leave it as a free parameter, because this freedom happens to be convenient for some manipulations. Thus, from now on we shall use the general expression (9) for R_i, unless stated otherwise.

3. Sum rules

3.1. Techniques

We shall derive old and new sum rules by using combinations of the following procedures:

(a) Use of equation (10) to express raising operators $R(w_i)$ as sums over $R(x_j)$.

(b) Expressing powers of $R(w_i)$ (to shorten the notation we omit α and β) in terms of powers of $R(x_j)$ by using the multinomial expansion:

$$\begin{aligned} R^n(w_i) &= (O_{i1}R(x_1) + O_{i2}R(x_2) + \cdots + O_{iN}R(x_N))^n \qquad i = 1, \ldots, N, \\ &= \sum_{|p|=n} \left\{ \frac{n!}{p_1!p_2!\cdots p_N!} \prod_{j=1}^{N} (O_{ij}R(x_j))^{p_j} \right\}, \\ &\text{where} \qquad |p| \equiv p_1 + \cdots + p_N. \end{aligned} \tag{14}$$

This expansion is possible, since the $R(x_i)$ commute with each other, $[R(x_i), R(x_j)] = 0$. For $N = 2$ equation (14) is a binomial expansion,

$$R^n(O_{i1}x + O_{i2}y) = \sum_{s=0}^{n} \binom{n}{s} O_{i1}^{n-s} O_{i2}^{s} R^{n-s}(x) R^s(y), \qquad i = 1, 2, \tag{15}$$

where $x \equiv x_1$ and $y \equiv x_2$.

(c) The $R(w_i)$ commute among themselves, because the $R(x_j)$ do. Therefore, by multiplying expansions of type (14) of *different* $R^{m_i}(w_i)$ and collecting the resulting powers of $R(x_j)$, we get homogeneous polynomials in $R(x_j)$,

$$\prod_{i=1}^{N} R^{m_i}(w_i) = \sum_{|p|=|m|} C_{[m][p]}(O) \prod_{j=1}^{N} R^{p_j}(x_j), \qquad \text{where} \qquad [p] \equiv [p_1, \ldots, p_N], \tag{16}$$

where $C_{[m],[p]}(O)$ are constants, which depend on O in a complicated way, in general. For $N = 2$ these expressions are still manageable, every element $O \in SO(2)$ can be written as follows:

$$\begin{aligned} &w_1 = cx - sy \qquad \text{and} \qquad w_2 = sx + cy, \\ &\text{where} \qquad c, s \in \mathbb{C}, \quad \text{with} \quad c^2 + s^2 = 1. \end{aligned} \tag{17}$$

Applying equation (15) to the transformation (17) gives

$$\begin{aligned} R^m(cx - sy)R^n(sx + cy) &= \sum_{p=0}^{m} \sum_{q=0}^{n} \binom{m}{p} \binom{n}{q} \\ &\quad \times (-1)^{m-p} c^{n+p-q} s^{m+q-p} R^{p+q}(x) R^{m+n-p-q}(y) \\ &\equiv \sum_{r=0}^{m+n} C_{[m,n],[r,m+n-r]}(c, s) R^r(x) R^{m+n-r}(y), \end{aligned} \tag{18}$$

so, by substituting $r = p + q$ in equation (18) we get

$$C_{[m,n],[r,m+n-r]}(c, s) = \sum_{q=0}^{\min(n,r)} \binom{m}{r-q}\binom{n}{q}(-1)^{m-r+q} c^{n+r-2q} s^{m+2q-r}. \tag{19}$$

(d) The inner product of the operator-valued vector $\mathbf{R}(x) := (R(x_1), R(x_2), \ldots, R(x_N))$ with itself is invariant under rotations. Hence,

$$\mathbf{R}\cdot\mathbf{R} = \sum_{i=1}^{N} R^2(w_i) = \sum_{j=1}^{N} R^2(x_j). \tag{20}$$

(e) Applying the operator equations (14), (16), (20), and their products to the constant ground function $g_0(x) \equiv 1$ will lead to relations among the multi-dimensional Hermite polynomials, that have shown usefulness in quantum optics [13].

3.2. New proofs of known sum rules

We shall first prove two known sum rules, using our operator techniques. These are usually proved by using the generating function, without reference to the symmetry involved. Here, we shall see that one is based on rotational symmetry and the other follows from translational symmetry.

3.2.1. The summation theorem

Proposition 1. *Let $w_i = (Ox)_i$ be defined as in equation (7). Then the following sum rule holds for the corresponding $\mathcal{GHP}$*

$$\tilde{H}_n(w_i) = \sum_{|p|=n} \left\{ \frac{n!}{p_1!p_2!\cdots p_N!} \prod_{j=1}^{N} (O_{ij})^{p_j} \tilde{H}_{p_j}(x_j) \right\} \tag{21}$$

Proof. Equation (21) follows immediately by applying the operator equality (14) to $H_0(w_i) = H_0(x_i) \equiv 1$. □

The sum rule (21) is known as the 'summation theorem' in the literature. It is usually written in the following cumbersome way [8, section 8.958], [14, p 254]:

$$\frac{1}{n!}\left(\sum_{k=1}^{N} a_k^2\right)^{\frac{n}{2}} H_n\left(\frac{\sum_{k=1}^{N} a_k x_k}{\sqrt{\sum_{k=1}^{N} a_k^2}}\right) = \sum_{p_1+p_2+\cdots+p_N=n} \prod_{j=1}^{N} \left\{ \frac{a_j^{p_j}}{p_j!} H_{p_j}(x_j) \right\}. \tag{22}$$

With hindsight, we easily recognize that the summation theorem is based on rotational symmetry, by noting that $O_{1k} \equiv a_k \big/ \sqrt{\sum_{k=1}^{N} a_k^2}$ for $k = 1, \ldots, N$ can be regarded, say, as the first row of an orthogonal matrix $O \in O(N)$.

For $N = 2$ and the special rotation:

$$\begin{pmatrix} r_- \\ r_+ \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{23}$$

equation (21) leads to the sum rule (24), due to C Runge [15].

Corollary 1. *The $\mathcal{GHP}$ satisfy the sum rule*

$$2^{n/2}\tilde{H}_n\left(\frac{x+y}{\sqrt{2}}\right) = \sum_{s=0}^{n}\binom{n}{s}\tilde{H}_s(x)\tilde{H}_{n-s}(y). \tag{24}$$

This sum rule also follows directly by applying the operator equation (15) to the ground function $g(x) = 1$ and then use the definition of the $\mathcal{GHP}$, equation (12).

3.2.2. A sum rule due to translation symmetry. We shall now prove another sum rule, which is usually derived [16] by using the generating function (A.13). Our derivation shows that the sum rule is due to translation symmetry:

Proposition 2. *The sum rule for the $\mathcal{GHP}$*

$$\tilde{H}_n(x+y) = \sum_{s=1}^{n}\binom{n}{s}(\alpha y)^{n-s}\tilde{H}_s(x). \tag{25}$$

holds.

Proof. Let us first assume that y is a variable which is independent of x. We shall relax this restriction later on. Then, if the differential operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial(x+y)}$ are applied to functions of $(x+y)$ only, one gets the same result,

$$\frac{\partial}{\partial x}f(x+y) = \frac{\partial}{\partial(x+y)}f(x+y). \tag{26}$$

Therefore, we get the following 'equivalence relation', denoted by $\simeq$, between the translated and the original raising operators, *provided that they both are applied to functions of* $(x+y)$,

$$R(\alpha,\beta;x+y) \simeq \alpha(x+y) - \beta\frac{\partial}{\partial x} = \alpha y + R(\alpha,\beta;x). \tag{27}$$

The rest of the proof follows by induction. Applying both sides of equation (27) to 1 yields $\alpha(x+y)$, which is certainly a function of $x+y$. Each additional application of both sides of equation (27) on the previously produced function of $x+y$ produces, in turn, a new function, which depends on $x+y$ only. Thus, by applying powers of both sides of equation (27) to $f(x+y) = f(x) = 1$ or to any function $f = f(x+y)$, will lead to the same result.

Finally, by applying powers of equation (27) to $\tilde{H}_0(x+y) = \tilde{H}_0(x) = 1$, and using the binomial expansion, we immediately get the sum rule (25).

Note that since sum rule (25) is an equality between polynomials, it holds identically for *any* x and y, even if y is a function of x, $y = h(x)$. Therefore, we can now forget about our earlier assumption, that y is a variable which is independent of x. □

The sum rule (25) is quite useful, it can be used to calculate the integral (30), below, used for evaluating the Wigner function [17]

$$W(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty}\Psi^*(x+y)\Psi(x-y)\,\mathrm{e}^{2\mathrm{i}xy}\,\mathrm{d}y = \frac{1}{\pi}\sum_{m,n=0}^{\infty}c_m^*c_nW_{m,n}(x,y) \tag{28}$$

for $\Psi(x) = \sum_{n=0}^{\infty}c_n\psi_n(x)$ where $\psi_n(x)$ is the wavefunction of the HO, given in equation (81) and

$$W_{mn}(x,p) = \frac{1}{\pi}\int_{-\infty}^{\infty}\psi_m^*(x+y)\psi_n(x-y)\,\mathrm{e}^{2\mathrm{i}py}\,\mathrm{d}y. \tag{29}$$

As an illustration we show how the integral (30) can be calculated by using the sum rule (25).

Corollary 2. *For $y, z \in \mathbb{C}$, we have [8, section 7.377]:*

$$I_{mn}(y,z) \equiv \int_{-\infty}^{\infty} H_m(x+y) H_n(x+z)\,\mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi} m! 2^n z^{n-m} L_m^{n-m}(-2yz), \quad \text{for} \quad m \leqslant n, \tag{30}$$

where $L_m^{\alpha}(x)$ are the associated Laguerre polynomials.

Proof. Substituting the expansion (25) for the *standard* Hermite polynomials ($\alpha = 2$) into the integrand of (30), we get

$$\begin{aligned} I_{mn}(y,z) &= \sum_{r=0}^{m}\sum_{s=0}^{n} \binom{m}{r}\binom{n}{s} (2y)^{m-r}(2z)^{n-s} \int_{-\infty}^{\infty} H_{m-r}(x) H_{n-s}(x)\,\mathrm{e}^{-x^2}\,\mathrm{d}x \\ &= \sum_{r=0}^{m}\sum_{s=0}^{n} \binom{m}{r}\binom{n}{n-s} (2z)^{s-r}(4yz)^r \sqrt{\pi} 2^{m-r}(m-r)!\delta_{m-r,n-s} \\ &= \sqrt{\pi} m! 2^m (2z)^{n-m} \sum_{r=0}^{m} \binom{n}{m-r} \frac{(2yz)^r}{r!} = \sqrt{\pi} m! 2^n z^{n-m} L_m^{n-m}(-2yz), \end{aligned} \tag{31}$$

where we used the orthogonality relation (56) and then the power expansion for the associated Laguerre polynomials [8, section 8.970]. □

3.3. New sum rules

We now prove the factorization sum rule (5) and a few others, by using combinations of the above techniques.

3.3.1. The factorization sum rule and corollaries.

Proposition 3 (generalized factorization sum rule)**.** *The $\mathcal{GHP}$ satisfy the following sum rule, for $c, s \in \mathbb{C}$ and $c^2 + s^2 = 1$:*

$$\tilde{H}_m(cx - sy)\tilde{H}_n(sx + cy) = \sum_{r=0}^{m+n} C_{[m,n],[r,m+n-r]}(c,s)\tilde{H}_r(x)\tilde{H}_{m+n-r}(y), \tag{32}$$

where the coefficients C are given in equation (19).

Proof. These sum rules follow immediately by applying the operator identities (18) to the ground function $g(x, y) \equiv 1$. □

The conjectured sum rule (33) is a special case of (32), where $m = n$ and $c = s = 1/\sqrt{2}$. But instead of calculating the coefficients C (equation (19)) for this special case, it is easier to give a direct proof of (33).

Proposition 4 (Factorization sum rule)**.** *The $\mathcal{GHP}$ satisfy the following sum rule:*

$$2^n \tilde{H}_n\left(\frac{x-y}{\sqrt{2}}\right)\tilde{H}_n\left(\frac{x+y}{\sqrt{2}}\right) = \sum_{s=1}^{n}\binom{n}{s}(-1)^s \tilde{H}_{2s}(y)\tilde{H}_{2n-2s}(x) \tag{33}$$

Proof. For the transformation (23), we have $R(r_\pm) = (R(x) \pm R(y))/\sqrt{2}$, and therefore

$$(2R(r_-)R(r_+))^n = (R^2(x) - R^2(y))^n = \sum_{s=1}^{n}\binom{n}{s}(-1)^s R^{2s}(y)R^{2n-2s}(x), \tag{34}$$

which immediately leads to equation (33) by applying both sides of equation (34) to 1. □

By comparing the expansion (32) for $c = s = 1/\sqrt{2}$ and $m = n$ with (33), we immediately get the following sum rule for the binomial coefficients:

Corollary 3.

$$2^n C_{[n,n],[r,2n-r]}(c = s = 1/\sqrt{2}) = (-1)^n \sum_{q=0}^{\min(n,r)} \binom{n}{r-q}\binom{n}{q}(-1)^q$$
$$= \begin{cases} 0 & \text{for} \quad r \text{ odd} \\ (-1)^{r/2}\binom{n}{r/2} & \text{for} \quad r \text{ even.} \end{cases}$$

From sum rule (33) we can also derive easily many other sum rules for Hermite polynomials. Here are two further examples:

Corollary 4. *For* $r_\pm := (x \pm y)/\sqrt{2}$*, we get*

$$2^{n+1/2}\tilde{H}_{n+1}(r_+)\tilde{H}_n(r_-) = \sum_{s=1}^{n}\binom{n}{s}(-1)^s[\tilde{H}_{2s}(y)\tilde{H}_{2n+1-2s}(x) + \tilde{H}_{2s+1}(y)\tilde{H}_{2n-2s}(x)], \qquad (36)$$

and

$$2^{n-3/2}[\tilde{H}_{n-1}(r_+)\tilde{H}_n(r_-) - \tilde{H}_n(r_+)\tilde{H}_{n-1}(r_-)]$$

Note that equation (40) could also be proved by using the sum rule (1) and equating both sides to a Laguerre polynomial. Similarly, the *N*-dimensional version, sum rule (38), can also be proved by using the *N*-dimensional version of (2) [8, section 8.977]. However, our proof, which is based on the operator techniques, is more direct and does not require any borrowed formulas. It also exhibits the rotational invariance, which is the basis of the above sum rules.

By combining the operator equalities which led to the previous two propositions, we now get a new sum rule.

Proposition 6. *The $\mathcal{GHP}$ satisfy the sum rule*

$$\sum_{s=1}^{n}\binom{n}{s}\tilde{H}_{n+2s}(r_+)\tilde{H}_{3n-2s}(r_-) = 2^{-n}\sum_{s=1}^{n}\binom{n}{s}(-1)^s\tilde{H}_{4s}(y)\tilde{H}_{4n-4s}(x). \quad (41)$$

Proof. Applying equation (34) on $[R^2(r_+) + R^2(r_-)]^n = [R^2(y) + R^2(x)]^n$, we get

$$[2^n R^n(r_+)R^n(r_-)][R^2(r_+) + R^2(r_-)]^n = [R^4(x) - R^4(y)]^n, \quad (42)$$

which, by using the expansion 39 for $N = 2$, gives

$$2^n\sum_{s=1}^{n}\binom{n}{s}R^{n+2s}(r_+)R^{3n-2s}(r_-) = \sum_{s=1}^{n}\binom{n}{s}(-1)^s R^{4s}(y)R^{4n-4s}(x), \quad (43)$$

that leads to equation (41). □

As a final illustration, we prove the following interesting sum rule for $N = 3$, which shows that an amazing variety of sum rules can be derived by using the above techniques:

Proposition 7.

$$6^{n/2}\tilde{H}_n\left(\frac{x-y}{\sqrt{2}}\right)\tilde{H}_n\left(\frac{x+y+z}{\sqrt{3}}\right) = \sum_{p_1+\cdots+p_4=n}\frac{n!(-1)^{p_2+p_4}}{p_1!p_2!p_3!p_4!}\tilde{H}_{2p_1+p_3}(x)\tilde{H}_{2p_2+p_4}(y)\tilde{H}_{p_3+p_4}(z). \quad (44)$$

Proof. For the orthogonal transformation:

$$w_1 = \frac{1}{\sqrt{2}}(x_1 - x_2), \qquad w_2 = \frac{1}{\sqrt{3}}(x_1 + x_2 + x_3), \qquad w_3 = \frac{1}{\sqrt{6}}(x_1 + x_2 - 2x_3), \quad (45)$$

we have

$$\sqrt{6}R(w_1)R(w_2) = (R_1 - R_2)(R_1 + R_2 + R_3) = R_1^2 - R_2^2 + R_1R_3 - R_2R_3, \quad (46)$$

where $R_i := R(x_i)$. The multinomial expansion of the *n*th power of (46) gives

$$6^{n/2}R^n(w_1)R^n(w_2) = \sum_{p_1+\cdots+p_4=n}\frac{n!}{p_1!p_2!p_3!p_4!}R_1^{2p_1}\left(-R_2^2\right)^{p_2}(R_1R_3)^{p_3}(-R_2R_3)^{p_4}$$
$$= \sum_{p_1+\cdots+p_4=n}\frac{n!}{p_1!p_2!p_3!p_4!}(-1)^{p_2+p_4}R_1^{2p_1+p_3}R_2^{2p_2+p_4}R_3^{p_3+p_4}. \quad (47)$$

By applying this operator equation to $g_0(x) \equiv 1$, we immediately get the sum rule (44). □

3.4. Sum rules with special arguments

Most of our sum rules are consequences of rotational invariance symmetries. For this reason they depend on two or more variables. But by substituting special arguments, such as $y = x$, in our 'primary' rotational sum rules, we can get a wide variety of more specific sum rules, *with a reduced number of variables*. Such sum rules may be called *derived or secondary sum rules*. If we look directly at such derived sum rules we may not recognize their very rotational origin.

For example, the sum rule

Corollary 6.

$$H_n(x)H_n(x) = 2^{-n}\sum_{s=1}^{n}\binom{n}{s}\frac{(2s)!}{s!}H_{2n-2s}(\sqrt{2}x) \tag{48}$$

is a special case of sum rule (33), where one sets $y = 0$, *changes* $x/\sqrt{2} \to x$ *and substitutes* $H_{2s}(0) = (-1)^s(2s)!/s!$. *Here we use the standard Hermite polynomials* $H_n(x) \equiv \tilde{H}(2, 1; x)$, *for easy comparison:*

At first glance, the sum rule (48) looks like a special case (for $m = n$*) of the sum rule in one variable,*

$$H_m(x)H_n(x) = \sum_{s=0}^{\min(m,n)} 2^s s!\binom{m}{s}\binom{n}{s}H_{m+n-2s}(x) \tag{49}$$

due to Feldheim [14, 18, p 255]. A closer look shows that the sum rules (48) and (49) are different, because the argument of the H_k *on the rhs of (48) is* $\sqrt{2}x$ *whereas that of (49) is* x. *Apparently Feldheim sum rule does not depend on rotational symmetry. It has been proved by a variety of methods, including by induction [19].*

By specializing sum rule (49) to $m = n$ *and equating it to (48), we get the following amusing relation, involving Hermite polynomials of different arguments,* $\sqrt{2}x$ *and* x*:*

Corollary 7.

$$H_n^2(x) = 2^{-n}\sum_{s=1}^{n}\binom{n}{s}\frac{(2s)!}{s!}H_{2n-2s}(\sqrt{2}x) = \sum_{s=0}^{n}2^s s!\binom{n}{s}\binom{n}{s}H_{2n-2s}(x). \tag{50}$$

As a final illustration of using special arguments, we derive two secondary sum rules from (44):

Corollary 8.

$$6^{n/2}\tilde{H}_n\left(\frac{x-y}{\sqrt{2}}\right)\tilde{H}_n\left(\frac{x}{\sqrt{3}}\right) = \sum_{p_1+\cdots+p_4=n}\frac{n!(-1)^{p_2+p_3}}{p_1!p_2!p_3!p_4!}\tilde{H}_{2p_1+p_3}(x)\tilde{H}_{2p_2+p_4}(y)\tilde{H}_{p_3+p_4}(y) \tag{51}$$

$$6H_2\left(\frac{2x}{\sqrt{3}}\right) = -H_4(x) + H_2^2(x) + 2H_2(x) - 2H_1^2(x). \tag{52}$$

Proof. Sum rule (51) follows immediately from (44) by setting $z = -y$. Equation (52) follows by setting $z = 0$ and $x = y$ in equation (44), and using, $H_{2n+1}(0) = 0$ and $H_{2n}(0) = (-1)^n(2n)!/n!$. □

Primary sum rules and some of the secondary ones, such as (51), are easily recognizable as rotational sum rules, because the sums on the indices of the Hermite polynomials are the same for all terms. In contrast, the origin of sum rules such as (48) and (52) is difficult to recognize as being due to $O(N)$ symmetry, because the index balance gets lost when the Hermite polynomials with zero arguments are replaced by the constant factors, $H_{2s}(0) = (-1)^s(2s)!/s!$. For instance, in equation (52) the terms involving H_4 and H_2^2 have a total index 4, whereas the terms H_2 and H_1^2 have a total index 2.

3.5. General expansions and reducible representation of the $SO(N)$ group

Most of the sum rules that we have so far proved are, essentially, special cases of the following general result:

Proposition 8. *Let $w_i := (Ox)_i$. Then, every product of $\tilde{H}_{m_i}(w_i)$ of different indices i can be expanded linearly into similar products of $\tilde{H}_{p_i}(x_i)$:*

$$\prod_{i=1}^{N} \tilde{H}_{m_i}(w_i) = \sum_{|p|=|m|} C_{[m],[p]}(O) \prod_{i=1}^{N} \tilde{H}_{p_i}(x_i), \qquad \text{where} \qquad [p] \equiv [p_1, \ldots, p_N]. \tag{53}$$

Proof. This follows easily by applying (16) to $g(x) \equiv 1$. □

The expansion (53) may appear self-evident, but it is not actually, since it is not generally valid for unitary transformations, as we can verify from the following simple example. The transformation $w_1 = \mathrm{i}x$ and $w_2 = -\mathrm{i}y$ is unitary. But $H_2(w_1) = 4w_1^2 - 2 = -4x^2 - 2$ cannot be expanded linearly in terms of the $n = 2$ basis, $\{H_2(x), H_1(x)H_1(y), H_2(y)\}$.

In fact, if we normalize the above products properly, as follows,

$$h_{[p]}^n(x) := (\alpha\beta)^{-n/2} \prod_{i=1}^{N} \frac{\tilde{H}_{p_i}(x_i)}{\sqrt{p_i!}} \qquad \text{where} \qquad [p] \equiv [p_1, \ldots, p_N] \qquad \text{and} \qquad n := |p|, \tag{54}$$

we obtain a basis for a (reducible) *orthogonal* representation of the complex $SO(N)$ group.

Proposition 9. *The expansion coefficients $D_{[q],[p]}^n(O)$ in*

$$h_{[p]}^n(O^{-1}x) := \sum_{|q|=n} h_{[q]}^n(x) D_{[q],[p]}^n(O), \qquad \text{where} \qquad n := |p|, \tag{55}$$

form a (reducible) orthogonal representation of $O \in SO(N)$.

Proof. Here, we shall prove the above proposition for *real* $SO(N)$. The proof for the complex $SO(N)$ is given in [20]. The $\mathcal{GHP}$ are orthogonal with respect to the following scalar product:

$$\int_{-\infty}^{\infty} \tilde{H}_m(x)\tilde{H}_n(x) \exp[-\alpha x^2/(2\beta)]\,\mathrm{d}x = \left(\frac{2\beta\pi}{\alpha}\right)^{1/2} (\alpha\beta)^n n!\delta_{mn}, \qquad \text{for} \quad \alpha/\beta > 0. \tag{56}$$

Therefore, the product of functions (54) form an orthonormal basis for each $N \geqslant 1$, relative to the scalar product

$$\langle h_{[p]}^n | h_{[q]}^m \rangle := \int h_{[p]}^n(x) h_{[q]}^m(x)\,\mathrm{d}\mu(x) = \prod_{i=1}^{N} \delta_{p_i,q_i}, \tag{57}$$

where

$$\mathrm{d}\mu(x) := \left(\frac{\alpha}{2\beta\pi}\right)^{N/2} \exp\left[-\alpha\left(x_1^2 + \cdots + x_N^2\right)/2\beta\right]\mathrm{d}^N x, \qquad (\text{for} \quad \alpha/\beta > 0), \tag{58}$$

is an invariant measure. Consequently, the transformed functions $h^n_{[p]}(O^{-1}x)$ are also orthonormal relatively to the same scalar product. Therefore, the matrix $D(O)$ must be orthogonal, because it transforms one orthonormal basis into another. □

As a consequence of the above orthogonality we can prove many relations involving the binomial coefficients. In particular, for $N = 2$, we get the following amusing relation:

Corollary 9. *The relation*

$$\sum_{r=0}^{m+n} \frac{r!(m+n-r)!}{m!n!} \left[\sum_{q=0}^{\min(n,r)} \binom{m}{r-q}\binom{n}{q}(-1)^{m-r+q} c^{n+r-2q} s^{m+2q-r}\right]^2 = 1 \tag{59}$$

holds identically for $c, s \in \mathbb{C}$, and $c^2 + s^2 = 1$.

Proof. By noting the normalization factors in the expression (54) of the $h^{m+n}_{[m,n]}$, we get from equation (32)

$$D_{[r,m+n-r],[m,n]}(O) = \sqrt{\frac{r!(m+n-r)!}{m!n!}}\, C_{[m,n],[r,m+n-r]}(O). \tag{60}$$

The sum rule (59) follows by substituting the expression (19) for the C-coefficients into the orthogonality condition

$$\sum_{r=0}^{m+n} D^2_{[r,m+n-r],[m,n]}(O) = 1. \tag{61}$$

□

For $\alpha/\beta > 0$ the products of the functions $h^n_{[p]}(x)$ with the Gaussian factor $\exp[-\alpha x^2/(4\beta)]$ yield the wavefunctions of the isotropic HO in N-dimensions, with total energy $E_n = (n + N/2)\hbar\omega$. Thus, the transformation (55) corresponds to the mixing of the degenerate energy states under orthogonal transformations. Moreover, our arguments show that these transformations:

(1) are also valid for complex rotations, for the heat polynomials, $\alpha/\beta < 0$, and for the simple powers, $\beta = 0$;
(2) give explicit procedure to calculate the representation matrices $D(O)$ of equation (55).

4. The *sl*(2,R) algebra of the Laguerre polynomials

In this section we construct raising and lowering operators K_+ and K_- for Laguerre polynomials $L_n(z)$, by using the corresponding operators R and L for the Hermite polynomials.

4.1. Constructing the raising operator K_+ for the Laguerre polynomials

The following construction was inspired by equation (1). By applying

$$\tilde{K}_+ := -\tfrac{1}{4}[R^2(2,1;x) + R^2(2,1;y)] \tag{62}$$

to the identity function $g_0(x) = 1$, we create the rhs of equation (1), and with it the Laguerre polynomial L_n:

$$\tilde{K}_+^n \cdot 1 = \frac{1}{(-4)^n} \sum_{s=1}^{n} \binom{n}{s} H_{2n-2s}(x) H_{2s}(y) = n! L_n(x^2 + y^2). \tag{63}$$

This means that K_+ acts as a raising operator for the Laguerre polynomials, if it is applied to the constant function $L_0(r^2) := 1$,

$$\tilde{K}_+^n \cdot 1 = n! L_n(r^2), \qquad \text{so that} \qquad \tilde{K}_+ L_n(r^2) = (n+1) L_{n+1}(r^2). \tag{64}$$

Since the $L_n(r^2)$ are functions of the variable $z := r^2 = x^2 + y^2$, it is desirable to express the operator $\tilde{K}_+$ as a function of the variable z only. To do this, we use the commutation relations $\frac{\partial}{\partial x_k} x_k = x_k \frac{\partial}{\partial x_k} + 1$, and get

$$\begin{aligned} \tilde{K}_+ &\equiv -\frac{1}{4}[R^2(2,1;x_1) + R^2(2,1;x_2)] = -\sum_{k=1}^{2} \left(x_k - \frac{1}{2}\frac{\partial}{\partial x_k} \right)^2 \\ &= \sum_{k=1}^{2} \left(-\frac{1}{4}\frac{\partial^2}{\partial x_k^2} + x_k \frac{\partial}{\partial x_k} - x_k^2 + \frac{1}{2} \right) = -\frac{1}{4}\Delta + r\frac{\partial}{\partial r} - r^2 + 1. \end{aligned} \tag{65}$$

The two-dimensional Laplacian Δ is composed of a radial part and an angular momentum part: $\Delta = \Delta_r - L_z^2/r^2$. In the following, we shall neglect the angular momentum term, since we shall apply the operator $\tilde{K}_+$ to cylindrically symmetric functions only $f(r^2)$, on which L_z^2 gives 0.

We now rewrite Δ_r in terms of $z = r^2$, by using the identity $D := \partial/\partial z = \partial/\partial r^2 = (1/(2r))\partial/\partial r$:

$$\begin{aligned} \Delta_r &= \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} \right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r^2 \left(\frac{1}{r}\frac{\partial}{\partial r} \right) \right) = 4DzD \\ &= 4(D + zD^2). \end{aligned} \tag{66}$$

Finally, by substituting this identity into equation (65) and neglecting the L_z^2/r^2 term, we get the desired expression for the raising operator K_+,

$$K_+(z) = \tilde{K}_+ - \frac{L_z^2}{4r^2} = -zD^2 - D + 2zD - z + 1. \tag{67}$$

4.2. The algebra generated by K_+ and K_- operators

Defining the differential operator

$$K_-(z) := -zD^2 - D = -\tfrac{1}{4}\Delta_r, \tag{68}$$

and by calculating the commutation with K_+, we get the new operator

$$K_0 := \tfrac{1}{2}[K_-, K_+] = \tfrac{1}{2}(K_- + K_+ + z) = -zD^2 - D + zD + 1/2. \tag{69}$$

Since the three operators K_0, K_- and K_+ satisfy the commutation relations $[K_-, K_+] = 2K_0, [K_0, K_+] = K_+, [K_0, K_-] = -K_-$, they are promptly recognized belonging to the $sl(2, R) \simeq su(1, 1)$ algebra [3, 12]. Thus, the above K operators provide a representation of $sl(2, R)$ in terms of differential operators. This algebra has many applications in physics, and Wybourne devoted a whole chapter to these applications [12]. Moreover, this algebra has the following two-dimensional representation

$$K_+ \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad K_- \to -\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad K_0 \to \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{70}$$

By comparing equation (69) with the well-known defining equation for the Laguerre polynomials [8, section 8.979] $(zD^2 + (1-z)D + n)L_n(z) = 0$, we can easily see that these polynomials are eigenfunctions of K_0,

$$K_0 L_n = (n + 1/2)L_n. \tag{71}$$

Finally, using (71) and the commutation relation $[K_0, K_-] = -K_-$ we verify that K_- is a *lowering operator for Laguerre polynomials*:

$$K_-(z)L_n(z) = nL_{n-1}(z). \tag{72}$$

4.3. An infinite-dimensional representation of $sl(2, R)$ *group operators*

The relations

$$K_+ L_n = (n+1)L_{n+1}, \qquad \text{and} \qquad K_- L_n = nL_{n-1}, \tag{73}$$

suggest the following infinite-dimensional matrix representation of $sl(2, R)$, where $k_+ = k_-^\dagger$:

$$k_+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & \ddots \end{pmatrix}, \qquad k_- = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & \ddots \end{pmatrix}. \tag{74}$$

The commutator of these two matrices yields the representation of K_0:

$$k_0 = \frac{1}{2}[k_-, k_+] = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & \ddots \end{pmatrix}, \tag{75}$$

which is a diagonal matrix, as expected from (71). This unitary representation is obtained by mapping the $L_n(x)$ onto the following vector basis:

$$L_0(x) \mapsto |0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \qquad L_1(x) \mapsto |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \qquad L_2(x) \mapsto |2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \text{etc.} \tag{76}$$

5. Applications of the sum rules and the $\mathcal{GHP}$

5.1. Evaluating Wigner functions using the factorization sum rule

As an interesting application of the factorization sum rule (40), we shall first evaluate the integral

$$\int_{-\infty}^{\infty} H_n(x+y)H_n(x-y)\,\mathrm{e}^{-y^2}\,\mathrm{e}^{2\mathrm{i}py}\,\mathrm{d}y$$

and then use it to calculate the diagonal Wigner function W_{nn} of (29), associated with the HO elementary projector $|n\rangle\langle n|$:

Proposition 10.

$$I_n(x, p) \equiv \int_{-\infty}^{\infty} H_n(x+y)H_n(x-y)\,\mathrm{e}^{-y}$$

5.2. Some applications of the $\mathcal{GHP}$

The Hermite polynomials are quite familiar to physicists, since they enter in the wavefunctions of the HO [25]. Below, we give three applications of the $\mathcal{GHP}$ which are less familiar.

(1) *Heat equation.* By differentiating the explicit expression (A.3), in appendix A, with respect to β and using the recursion formulas for the Hermite coefficients,

$$s\begin{bmatrix} n \\ s \end{bmatrix} = n(n-1)\begin{bmatrix} n-2 \\ s \end{bmatrix}, \tag{84}$$

we get

$$\frac{\partial}{\partial\beta}\tilde{H}_n(\alpha,\beta;x) = -n(n-1)\frac{\alpha}{2}\tilde{H}_{n-2}(\alpha,\beta;x) = -\frac{1}{2\alpha}\frac{\partial^2}{\partial x^2}\tilde{H}_n(\alpha,\beta;x), \tag{85}$$

where we also used equation (B.1) to obtain the last equality. Equation (85) immediately shows that the *heat polynomials* $\Phi_n(t,x) = \tilde{H}_n(1,-2t;x)$ are solutions of the heat equation [10]

$$\frac{\partial}{\partial t}u(t;x) = \frac{\partial^2}{\partial x^2}u(t;x), \qquad \text{with the initial conditions} \quad u(0;x) = x^n. \tag{86}$$

Note that the heat polynomials, which are special $\mathcal{GHP}$, are much more suitable for describing the solutions of (86), than the standard Hermite polynomials, because $\tilde{H}_n(1,-2t;x)$ remains finite for $t=0$, whereas using the standard Hermite polynomials instead, would lead for $t\to 0$ to Hermite polynomials with infinite arguments $H_n(\mathrm{i}x/(2\sqrt{t}))$, multiplied by a vanishing multiplicative factors $(-i\sqrt{t})^n$ (see (A.13)).

(2) *Generalized Poisson distributions.* From expression (A.3), we see that the $\mathcal{GHP}$ are positive functions for negative β and positive α and x. Therefore, the functions

$$P_n(\gamma;x) := \frac{1}{n!}\Phi_n(\gamma,x) = \frac{1}{n!}\tilde{H}_n(1,-2\gamma;x)\,\mathrm{e}^{-x-\gamma} \tag{87}$$

can be viewed as probability densities, because $P_n \geqslant 0$ and

$$\sum_{n=0}^{\infty} P_n(\gamma;x) = 1. \tag{88}$$

This sum follows by substituting $t=\alpha=1$ and $\beta=-2\gamma$ in equation (A.16) for the generating function. In the limit $\gamma\to 0$, the above distribution becomes a Poisson distribution,

$$\lim_{\gamma\to 0} P_n(\gamma;x) = \frac{1}{n!}x^n\,\mathrm{e}^{-x}.$$

(3) *Squeezed states and pseudo-diffusion equation.* Another probability density distribution, now in the (q,p) phase space, is obtained by squaring the amplitudes $\langle p,q;\lambda|n\rangle$, which is the HO basis state $|n\rangle$ in the squeezed states representation $|p,q;\lambda\rangle$, where λ is related to the *real* squeezing parameter y, by $\lambda\equiv \mathrm{e}^{2y}$, so that $\lambda=1$ (or $y=0$) corresponds to the common (unsqueezed) coherent state. Using our operator techniques, we were able to calculate the above amplitudes directly in terms of $\mathcal{GHP}$ [26], so that

$$P_n(p,q;\lambda) := |\langle p,q;\lambda|n\rangle|^2 = \frac{2\sqrt{\lambda}}{2^n n!(\lambda+1)}|\tilde{H}_n(2,\xi;w_\lambda)|^2 \exp\left[-\frac{\lambda q^2+p^2}{\lambda+1}\right],$$
$$n\geqslant 0, \tag{89}$$

where

$$\xi := \frac{\lambda^{1/2} - \lambda^{-1/2}}{\lambda^{1/2} + \lambda^{-1/2}}, \qquad \text{and} \qquad w_\lambda := \frac{\lambda^{1/2} q + i\lambda^{-1/2} p}{\lambda^{1/2} + \lambda^{-1/2}}. \tag{90}$$

These $P_n(p, q; \lambda)$ functions are probability distributions in the phase-space variables (q, p), because the squeezed states are normalized, $\langle p, q; \lambda | p, q; \lambda\rangle = 1$, for all λ. In the limit of zero squeezing $y \to 0$ (i.e. $\lambda \to 1$) we get $\xi \to 0$, and (89) converges to a Poisson distribution with parameter (mean value) $(q^2 + p^2)/2$,

$$\begin{aligned}\mathcal{P}_n\left(\frac{q^2+p^2}{2}\right) &:= P_n(p, q; 1) = \frac{1}{2^n n!}|\tilde{H}_n(2, 0; w_1)|^2 \exp\left[-\frac{q^2+p^2}{2}\right] \\ &= \frac{1}{n!}\left(\frac{q^2+p^2}{2}\right)^n \exp\left[-\frac{q^2+p^2}{2}\right], \qquad n \geqslant 0.\end{aligned} \tag{91}$$

The $\mathcal{GHP}$ has enabled us to obtain the $\xi = 0$ limit without effort, since

$$\lim_{\xi\to 0} \tilde{H}_n(2, \xi; w_\lambda) = \tilde{H}_n(2, 0; w_1) = (2w_1)^n = (q + ip)^n. \tag{92}$$

Alternatively, if we use in equation (89) the standard Hermite polynomials instead of the $\mathcal{GHP}$, we obtain by using the scaling relation (A.7), the more familiar expression [4, 5, 7]:

$$P_n(p, q; \lambda) := \frac{1}{n!\mu}\left(\frac{\nu}{2\mu}\right)^n \left|H_n\left(\frac{\beta}{\sqrt{2\mu\nu}}\right)\right|^2 \exp\left[-|\beta|^2 + \mathrm{Re}\left(\frac{\bar{\nu}}{\mu}\beta^2\right)\right], \qquad n \geqslant 0, \tag{93}$$

where

$$\begin{aligned}\nu &\equiv \sinh y = \frac{\lambda^{1/2} - \lambda^{-1/2}}{2}, \qquad \mu \equiv \cosh y = \frac{\lambda^{1/2} + \lambda^{-1/2}}{2}, \\ \beta &\equiv \frac{\lambda^{1/2} q + \mathrm{i}\lambda^{-1/2} p}{\sqrt{2}} = \sqrt{2}\mu w_\lambda.\end{aligned} \tag{94}$$

(In our case, ν is real, but we used $\bar{\nu}$ in (93), so that we get exactly the square of the absolute value of the expression for $\langle n|\beta\rangle$ given in equation (3.23) of [5].) Since $\nu \to 0$ for $y \to 0$, we see that the arguments of the standard Hermite polynomials in (93) become infinite in the limit $\xi = \nu/\mu \to 0$, and a careful study of products of infinite and vanishing quantities becomes necessary for getting the Poisson-distribution limit.

Thus, the use of the $\mathcal{GHP}$ enabled us to discover that the above probability densities satisfy the following 'pseudo diffusion equation' [26]:

$$\frac{\partial}{\partial\lambda} P_n(p, q; \lambda) = \frac{1}{4}\left(\frac{\partial^2}{\partial p^2} - \frac{1}{\lambda^2}\frac{\partial^2}{\partial q^2}\right) P_n(p, q; \lambda), \tag{95}$$

which describes the effect of squeezing as a pseudo diffusion of the probability densities $P_n(p, q; \lambda)$ in phase space.

6. Summary

We have introduced the generalized Hermite polynomials $\tilde{H}_n(\alpha, \beta; x)$, which have, as particular cases, the standard Hermite polynomials, the heat polynomials and also simple powers. We described their properties in the appendices. We discussed some applications of the $\mathcal{GHP}$ to the heat equation, generalized Poisson distributions and squeezed states. In particular, we showed that the $\mathcal{GHP}$ are especially useful for obtaining the $\beta \to 0$ limit.

Finding a new sum rule is an achievement, especially in such an old and well-established mathematical field, such as orthogonal polynomials. Here, we gave a general method which has the potential of proving and discovering numerous interesting sum rules for the $\mathcal{GHP}$. This method is based on the symmetry group $O(N)$ of complex orthogonal transformations. We described several techniques, and illustrated them by giving new proofs for old sum rules and by proving several new sum rules. We illustrated how some of the sum rules can be applied for evaluating certain integrals, which are useful for calculating the Wigner functions of the HO state functions.

We also used the raising and lowering operators for the Hermite polynomials to construct the operators, K_+, K_- and K_0, for the Laguerre polynomials. Their commutation relations lead to $sl(2, R)$ algebra, for which we gave an ∞-dimensional unitary representation.

As final remark it is worth noting that the $\mathcal{GHP}$ can be further generalized to polynomials of the form

$$H_n^{(r)}(\alpha, \beta; x) = \sum_{s=0}^{[n/r]} c_s^n x^{n-rs}, \qquad c_0^n \neq 0, \tag{96}$$

where r can be any positive integer, and not just 2. These 'r-polynomials' $H_n^{(r)}$ can be generated by using the raising operators $R_r := \alpha x - \beta D^{r-1}$. However, the above technique for obtaining sum rules cannot be applied for $r \neq 2$, since the generators R_r for $r \neq 2$ are not linear in the D operator and therefore they do not have simple transformation properties.

Acknowledgments

Work supported by FAPESP (SP, Brazil) contracts no 00/15084-5 and SSM acknowledges partial financial support from CNPq, Brasil.

Appendix A. Generalized Hermite polynomials, their symmetry properties and generating function

We define the *generalized Hermite polynomials* ($\mathcal{GHP}$) by means of the raising operators $R(\alpha, \beta, x)$ applied to the ground function $g_0(x) := 1$, as

$$\tilde{H}_n(\alpha, \beta; x) := R^n(\alpha, \beta; x) \cdot 1. \tag{A.1}$$

By successive application of R, we easily get

$$\begin{aligned} &\tilde{H}_0(x) = 1, && \tilde{H}_3(x) = \alpha^3 x^3 - 3\alpha^2 \beta x, \\ &\tilde{H}_1(x) = \alpha x, && \tilde{H}_4(x) = \alpha^4 x^4 - 6\alpha^3 \beta x^2 + 3\alpha^2 \beta^2, \\ &\tilde{H}_2(x) = \alpha^2 x^2 - \alpha\beta, && \tilde{H}_5(x) = \alpha^5 x^5 - 10\alpha^4 \beta x^3 + 15\alpha^3 \beta^2 x. \end{aligned} \tag{A.2}$$

These polynomials are special cases of the general expression for the $\mathcal{GHP}$

$$\tilde{H}_n(\alpha, \beta; x) := R^n(\alpha, \beta; x) \cdot 1 = \sum_{s=0}^{[n/2]} \begin{bmatrix} n \\ s \end{bmatrix} \left(-\frac{\alpha\beta}{2}\right)^s (\alpha x)^{n-2s}, \tag{A.3}$$

where

$$\begin{bmatrix} n \\ s \end{bmatrix} := \begin{bmatrix} n \\ s \end{bmatrix}, \tag{A.4}$$

that we call *Hermite coefficients*.

The $\mathcal{GHP}$ (A.3) has the following scaling properties:

$$\tilde{H}_n(\sigma^{-1}\alpha, \sigma\beta; \sigma x) = \tilde{H}_n(\alpha, \beta; x), \tag{A.5}$$

$$\tilde{H}_n(\sigma\alpha, \sigma\beta; x) = \sigma^n \tilde{H}_n(\alpha, \beta; x), \tag{A.6}$$

$$\tilde{H}_n(\alpha, \sigma^2\beta; \sigma x) = \sigma^n \tilde{H}_n(\alpha, \beta; x), \tag{A.7}$$

where again, the third relation follows from the first two. These scaling properties can also be obtained directly from the following symmetry properties of the raising operator (9):

$$R(\sigma^{-1}\alpha, \sigma\beta; \sigma x) = R(\alpha, \beta; x) = \frac{1}{\sigma} R(\alpha, \sigma^2\beta; \sigma x). \tag{A.8}$$

A.1. Particular cases

The $\mathcal{GHP}$ reduce to standard Hermite polynomials and to simple powers for special values of α and β:

$$\tilde{H}_n(2, 1; x) = H_n(x) = \sum_{s=0}^{[n/2]} \begin{bmatrix} n \\ s \end{bmatrix} (-1)^s (2x)^{n-2s}, \tag{A.9}$$

$$\tilde{H}_n(\alpha, 0; x) = (\alpha x)^n. \tag{A.10}$$

The scaling properties (A.5) and (A.6) allow us to relate the $\mathcal{GHP}$ to some standard definitions:

$$\tilde{H}_n(\alpha, \beta; x) = \left(\frac{\alpha\beta}{2}\right)^{n/2} \tilde{H}_n\left(2, 1; \sqrt{\frac{\alpha}{2\beta}}x\right) = \left(\frac{\alpha\beta}{2}\right)^{n/2} H_n\left(\sqrt{\frac{\alpha}{2\beta}}x\right), \tag{A.11}$$

$$\tilde{H}_n(1, 1; x) = \mathcal{H}e_n(x) = 2^{-n/2} H_n\left(\frac{x}{\sqrt{2}}\right), \tag{A.12}$$

$$\tilde{H}_n(1, -2t; x) = \Phi_n(t, x) = (-i\sqrt{t})^n H_n\left(\frac{ix}{2\sqrt{t}}\right), \qquad \text{for} \quad t > 0, \tag{A.13}$$

where $\Phi_n(t, x)$ are known as the *heat polynomials* [10, 11].

A.2. The generating function

By noting the Baker–Campbell–Hausdorff formula [3],

$$\mathrm{e}^{A+B} = \mathrm{e}^{\frac{1}{2}[A,B]}\, \mathrm{e}^A\, \mathrm{e}^B, \tag{A.14}$$

which holds if $[A, B]$ commutes with both A and B, we get

$$\mathrm{e}^{tR} = \mathrm{e}^{\alpha xt - t\beta D} = \mathrm{e}^{-\frac{\alpha\beta}{2}t^2}\, \mathrm{e}^{\alpha xt}\, \mathrm{e}^{-t\beta D} \qquad \text{for} \quad t \in \mathbb{C}. \tag{A.15}$$

We can now *derive* the *generating function of the $\mathcal{GHP}$*, by applying this operator equation to $g_0(x) \equiv 1$, and noting that $\mathrm{e}^{-t\beta D} \cdot 1 = 1$:

$$\mathrm{e}^{tR}.1 = \sum_{n=0}^{\infty} \frac{t^n}{n!} R^n \cdot 1 = \sum_{n=0}^{\infty} \frac{t^n}{n!} \tilde{H}_n(\alpha, \beta; x) = \mathrm{e}^{\alpha xt}\, \mathrm{e}^{-\frac{\alpha\beta}{2}t^2}. \tag{A.16}$$

A.3. Two proofs of the power expansion of the $\mathcal{GHP}$

The usual proof of the expression (A.3) for the standard Hermite polynomials [25] is obtained by expanding the lhs of (A.16) in powers of t, and equating it to its rhs.

Another proof of (A.3) follows by induction, by using the recursion relation (B.2) for the $\mathcal{GHP}$ and the following recursion formula for the Hermite coefficients:

$$\begin{bmatrix} n+1 \\ s \end{bmatrix} = \begin{bmatrix} n \\ s \end{bmatrix} + 2n \begin{bmatrix} n-1 \\ s+1 \end{bmatrix}. \tag{A.17}$$

Appendix B. The lowering operator *L*

It is easy to see that the operator (note that in [26] we normalized L differently, namely as D/β):

$$L := L(\alpha; x) := \frac{1}{\alpha} D, \qquad \text{where} \qquad D := \frac{\partial}{\partial x}, \tag{B.1}$$

satisfies the canonical commutation relation:

$$[L, R] := LR - RL = 1, \tag{B.2}$$

and therefore leads to

$$[L, R^n] = nR^{n-1}, \tag{B.3}$$

as can be proved induction. By applying (A.17) to 1, we get

$$L\tilde{H}_n(\alpha, \beta; x) = LR^n \cdot 1 = R^n L \cdot 1 + nR^{n-1} \cdot 1 = n\tilde{H}_{n-1}(\alpha, \beta; x), \tag{B.4}$$

where we used the fact that $L \cdot 1 = 0$.

Since L lowers the index of $\tilde{H}_n(\alpha, \beta; x)$, it will be called the *lowering operator* of the $\mathcal{GHP}$. Note that L is β-independent.

Applying $R = \alpha x - \beta D = \alpha x - \alpha\beta L$ to $\tilde{H}_n$, immediately leads to a recursion formula for the $\mathcal{GHP}$:

$$\tilde{H}_{n+1}(\alpha, \beta; x) = \alpha x \tilde{H}_n(\alpha, \beta; x) - \alpha\beta n \tilde{H}_{n-1}(\alpha, \beta; x). \tag{B.5}$$

Finally, we obtain the differential equation for the $\mathcal{GHP}$ by considering the eigenfunctions of the *number operator* $\mathcal{N} \equiv RL$. Since $\mathcal{N}\tilde{H}_n = n\tilde{H}_n$, we get

$$\left(-\frac{\beta}{\alpha} \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x} - n \right) \tilde{H}_n(\alpha, \beta; x) = 0. \tag{B.6}$$

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