Anyons and transmutation of statistics via a vacuum-induced Berry phase

Roberto M. Serra,[1] Angelo Carollo,[1,2] Marcelo França Santos,[1,3] and Vlatko Vedral[1]
[1]*Optics Section, The Blackett Laboratory, Imperial College, London SW7 2BZ, United Kingdom*
[2]*Centre for Quantum Computation, DAMTP, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom*
[3]*Departamento de Física, Universidade Federal de Minas Gerais, Belo Horizonte, 30161-970, MG, Brazil*
(Received 30 March 2004; published 14 October 2004)

We show that bosonic fields may present anyonic behavior when interacting with a fermion in a Jaynes-Cummings-like model. The proposal is accomplished via the interaction of a two-level system with two quantized modes of a harmonic oscillator; under suitable conditions, the system acquires a fractional geometric phase. A crucial role is played by the entanglement of the system eigenstates, which provides a two-dimensional confinement in the effective evolution of the system, leading to the anyonic behavior. For a particular choice of parameters, we show that it is possible to transmute the statistics of the system continually from fermions to bosons. We also present an experimental proposal, in an ion-trap setup, in which fractional statistical features can be generated, controlled, and measured.

DOI: 10.1103/PhysRevA.70.044102 PACS number(s): 03.65.Vf, 42.50.Vk, 05.30.Pr

Anyons are quasiparticles that exhibit fractional quantum statistics which arises when bosons or fermions are confined in a two-dimensional space [1,2]. Such excitations play a fundamental role in the physics of the fractional quantum Hall effect [3,4]. Beside the theoretical applications, anyonic systems are promising for the implementation of faulttolerant quantum computing [5].

In this paper, we show how typical bosons, specifically eigenstates of harmonic oscillators (number states), behave as fractional spin particles, so-called anyons, when coupled to a two-level system (a fermion) through a Jaynes-Cummings-like interaction.

We can interpret the origin of such an exotic behavior as a consequence of the entanglement capability of the interaction Hamiltonian (the entangled nature of the Hamiltonian eigenstates). When the coupled system is subjected to an adiabatic evolution that mixes the orthogonal modes of the harmonic oscillator, the entanglement provided by the Hamiltonian can be regarded as a two-dimensional constraint on one of the two subsystems, allowing for *bosonic excitations* of fields (photons or phonons) to behave like anyons. This fractional behavior is clearly manifested in the geometric phase [6,7] acquired by any eigenstate of the coupled system under a cyclic adiabatic evolution, and it is possible to relate the geometrical phase with the statistical factor [8].

The bosonic field behaves, in an analogous way, as a spin-1 particle when there is no interaction (in the sense that the geometric phase acquired during the evolution is the same as that acquired by a spin-1 particle in a magnetic field), to a spin-$1/4$ particle when the two systems are maximally entangled, going through all the intermediate spins depending on the specific degree of entanglement between the two subsystems. More specifically, it is possible to simulate anyons with $m/2, m/3, m/4, \ldots$ $(m=1,2,\ldots)$ statistics or even to transmute continually the statistics of the system from Fermi to Bose.

We also present a proposal, employing trapped ions, in which fractional statistical phases can be generated, manipulated, and tested. This proposal, in addition to its experimental interest, provides a clearer physical framework to interpret the presented theoretical results.

Initially, we consider an effective two-level (fermionic) system coupled nonlinearly to two quantized bosonic fields, through the so-called m-quantum Jaynes-Cummings model [9]. In the rotating-wave approximation, this Hamiltonian is given by $(\hbar=1)$

$$H = \nu a^{\dagger}a + \nu b^{\dagger}b + \frac{\omega}{2}\sigma_z + \lambda_m[\sigma_+(a)^m + \sigma_-(a^{\dagger})^m], \tag{1}$$

where $\sigma_z=|\uparrow\rangle\langle\uparrow|-|\downarrow\rangle\langle\downarrow|$, $\sigma_+=|\uparrow\rangle\langle\downarrow|$, and $\sigma_-=|\downarrow\rangle\langle\uparrow|$ are the usual Pauli pseudospin operators ($|\uparrow\rangle$ and $|\downarrow\rangle$ are the excited and ground states of the two-level system, respectively), λ_m is the effective m nonlinear coupling constant, and $a^{\dagger}$ (a) and $b^{\dagger}$ (b) are the creation (annihilation) operators of the bosonic fields with frequency ν.

In the interaction picture and in a rotating frame (through the unitary transformation $\exp[i\Delta_m\sigma_z t/2]$), H can be rewritten as

$$\mathbf{H} = \frac{\Delta_m}{2}\sigma_z + \lambda_m[\sigma_+(a)^m + \sigma_-(a^{\dagger})^m], \tag{2}$$

with $\Delta_m=\omega-m\nu$ being the effective detuning between the bosonic and fermionic systems. Field mode a is orthogonal to mode b and, initially, the two-level system interacts only with the former one.

The eigenstates of the joint system, associated to the eigenvalues $\pm\Lambda$ $(\Lambda=[(\Delta_m)^2/4+(\lambda_m)^2(n+m)!/n!]^{1/2})$, are described by

$$|\Psi^{\pm}_{n,n'}\rangle = (C_{\uparrow}|\uparrow\rangle|n\rangle_a \pm C_{\downarrow}|\downarrow\rangle|n+m\rangle_a)|n'\rangle_b, \tag{3}$$

where

$$C_{\uparrow} = \frac{\Lambda + \Delta_m/2}{\sqrt{2}\sqrt{\Lambda^2 + \Lambda\Delta_m/2}},$$

$$C_{\downarrow} = \frac{\lambda_m \sqrt{(n+m)!/n!}}{\sqrt{2}\sqrt{\Lambda^2 + \Lambda\Delta_m/2}},$$

and $|n\rangle_l$ $(l=a,b)$ is the Fock state of the bosonic field l.

Let us consider the case in which the Hamiltonian described by Eq. (1) is slowly transformed into a parametrized one, $\mathbf{H}(\theta,\phi)=U(\theta,\phi)\mathbf{H}U^{\dagger}(\theta,\phi)$. This transformation is achieved by externally driving the interacting systems, and if it is done adiabatically, i.e., slowly compared to the typical time scales of the problem, the eigenstates of the initial Hamiltonian follow the transformation, as defined in the adiabatic theorem. In the end of a closed cycle (i.e., θ and ϕ are cyclically varied and then brought back to their original value), each eigenstate goes back to the original one, except for a phase factor of geometrical nature $e^{i\gamma_{n,n'}}$ [6,7].

We are interested in the parametric transformation of Hamiltonian (1) obtained through the unitary operation

$$U(\theta,\phi) = \exp(-i\phi J_z)\exp(-i\theta J_y), \tag{4}$$

where $J_z=(a^{\dagger}a-b^{\dagger}b)/2$ and $J_y=i(ab^{\dagger}-a^{\dagger}b)/2$ are the Schwinger angular momentum operators. In this case, for each value of the parameters θ and ϕ, this transformation results in the two-level system interacting with a linear combination of the two bosonic fields, a and b [10,11].

A suitably slow variation of the parameters θ and ϕ results in the adiabatic evolution of the eigenstates $|\Psi^{\pm}_{n,n'}\rangle$ of the initial Hamiltonian $\mathbf{H}$ in Eq. (2). When θ and ϕ are eventually brought back to their initial values, $|\Psi^{\pm}_{n,n'}\rangle$ acquires a Berry phase, which is given by

$$\begin{aligned}\gamma_{n,n'} &= i\int_c d\phi d\theta \langle \Psi^{\pm}_{n,n'}|U^{\dagger}(\theta,\phi)\nabla_{\theta,\phi}U(\theta,\phi)|\Psi^{\pm}_{n,n'}\rangle \\ &= \Omega\langle\Psi^{\pm}_{n,n'}|J_z|\Psi^{\pm}_{n,n'}\rangle \\ &= \frac{\Omega}{2}\left[n-n'+\left(\frac{m}{2}\right)\frac{(\lambda_m)^2(n+m)!/n!}{\left(\Lambda^2+\Lambda\dfrac{\Delta_m}{2}\right)}\right],\end{aligned} \tag{5}$$

where Ω is the solid angle subtended by the cyclic path in Poincaré's sphere. Both eigenstates $|\Psi^{+}_{n,n'}\rangle$ and $|\Psi^{-}_{n,n'}\rangle$ acquire the same geometrical phase.

We are interested in the case of zero initial excitation in the bosonic fields $(n=n'=0)$, and the most interesting scenario is achieved when $\Delta_m=0$, i.e., the resonant interaction. In this regime, the harmonic oscillator and the two-level particle are exchanging m excitations and the geometric phase reduces to $\gamma_{0,0}=(m/4)\Omega$. Notice that, in this expression, each order of the nonlinear interaction in Eq. (1) is contributing with a factor of $\frac{1}{4}$ to the total geometric phase. The $\frac{1}{4}$ factor originates in the fermion-boson interaction as shown in [10], while the m factor is typical for a collective behavior. In this sense, the Hamiltonian (1) can be regarded as a multibody interaction, between the two-level system and m-bosonic particles.

The closed path described by the transformation (4) can be regarded as two consecutive exchanges of bosonic excitations in the two possible modes, which here play the role of two different spatial configurations of one anyonic particle. For example, initially the two-level system is exchanging m excitations with mode a. Then, for $\theta=\pi$, the exchange involves only mode b, and finally for $\theta=2\pi$, a complete cycle is done and the interaction is back to mode a. This is analogous to the case of one electron orbiting around a magnetic flux tube in the original Wilczek work [1]. The analogy with the classical rotation in real space is clearer in the trapped ions example presented later, where modes a and b can be chosen as nothing but spatial vibration modes of the trap (see Ref. [12] for the implementation of fractional dynamics in the context of Bose-Einstein condensation).

In analogy with the physical two-dimensional space, we exploit in our system the parametric space of the Hamiltonian. When the parameters of the Hamiltonian $\mathbf{H}(\theta,\phi)$ are changed and eventually returned to their original configuration, the wave function remains the same, except for the phase factor $e^{i\alpha 2\pi}$, where $\alpha=\gamma_{0,0}/2\pi$ is called the statistical factor. In a three-dimensional space, the rotation group satisfies a peculiar non-Abelian algebra, which allows only for the Bose $(\alpha=0)$ or Fermi statistics $(\alpha=1)$. On the other hand, if we restrict ourselves to two-dimensional rotations, the corresponding group can generate a broader class of (braid) statistics [13]. It turns out, in fact, that the impossibility of using rotations in a third dimension reduces the number of constraints on the symmetry of the wave function, allowing for fractional values of α.

We can interpret the eigenstate of the system described by Eq. (3) (an entangled state of the two-level system and the field) as a kind of two-dimensional confinement, in the sense that only the rotation J_z contributes for the Berry connection in Eq. (5), introducing the fractional behavior. To deepen our understanding of the role played by the entanglement of the eigenstates of the system (3), we will analyze the statistical factor α in different regimes, $\Delta_m \gg \lambda_m$, $\Delta_m \sim \lambda_m$ and $\Delta_m=0$. In general, the statistical factor is given by

$$\alpha = \frac{m}{4}\frac{\Omega}{2\pi}\frac{(\lambda_m)^2 m!}{\left(\Lambda^2+\Lambda\dfrac{\Delta_m}{2}\right)}, \tag{6}$$

and presents a dependence on the order m of the nonlinear interaction, on solid angle Ω, and also on the detuning Δ_m.

Let us consider the far off-resonance case, $\Delta_m \gg \lambda_m$, which means that the harmonic oscillators and the two-level system are not exchanging energy, i.e., the eigenstate of the Hamiltonian is completely separable. This is the most trivial case, where $\alpha=0$ and the excitations of the harmonic oscillators behave as bosons, as we should expect for a noninteracting system [10]. On the other hand, in the resonant case $(\Delta_m=0)$ the eigenstate of Hamiltonian (2) is maximally entangled and the statistical factor $\alpha=(m/4)(\Omega/2\pi)$. Therefore, the fractional features of the system's geometric phase depend crucially on the entangled form of the system's eigenstate. In Fig. 1(a), we show the ratio $\gamma_{0,0}[(m/4)\times(\Omega/2\pi)]^{-1}$ as a function of the detuning Δ_m, and in Fig. 1(b) we show the linear entropy of the reduced two-level system in state (3), $S_f=1-\mathrm{Tr}_f[(\mathrm{Tr}_b\rho)^2]$ (where f and b stand for fermionic and bosonic variables, respectively, and ρ

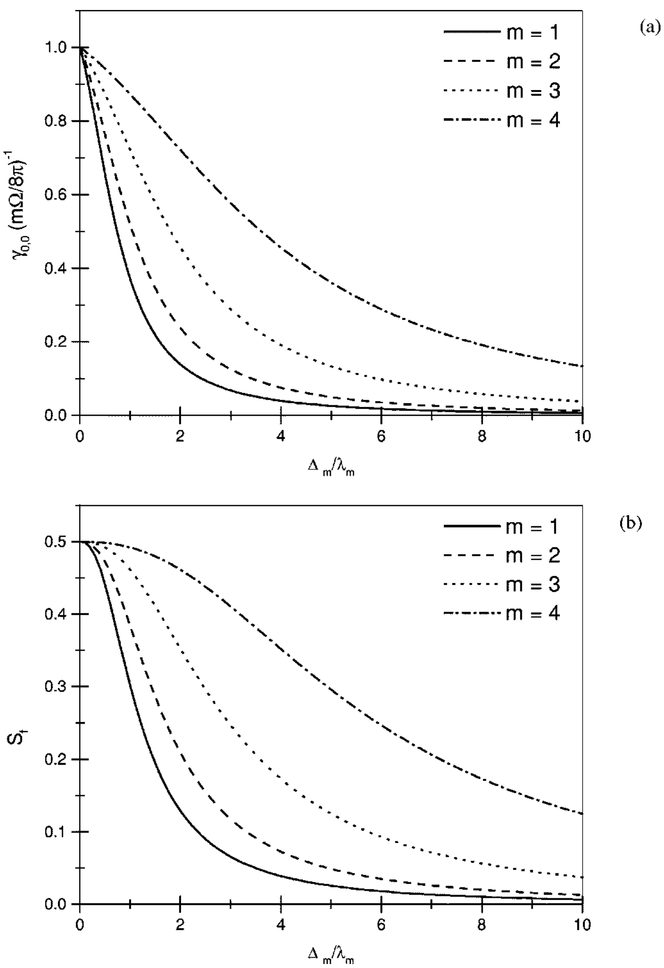

FIG. 1. (a) The fractional part of the geometrical phase $\{\gamma_{0,0}[(m/4)\Omega]^{-1}\}$ and (b) the linear entropy of the reduced two-level system in the state (3) as function of the effective detuning Δ_m.

$=|\Psi^{\pm}_{0,0}\rangle\langle\Psi^{\pm}_{0,0}|)$, as a function of the detuning Δ_m. Comparing these two figures, we can see that, when the detuning increases, the amount of entanglement decreases and consequently the fractional features of the system also decrease, until they disappear completely for separable eigenstates.

It is interesting to note that, if we vary θ,ϕ in a way that the cyclic path in Poincaré's sphere encloses the whole sphere ($\Omega=4\pi$), the statistical factor, for $\Delta_m=0$, turns out to be $\alpha=m/2$; if we choose a path that encloses $\frac{2}{3}$ of the sphere, $\alpha=m/3$; and so on. In this way, we can simulate anyons with $m/2, m/3, m/4,\ldots$ statistics. Let us consider the interesting particular situation (for $\Delta_m\neq 0$) in which we choose $m=2$ and vary the parameters θ,ϕ in order to obtain $\Omega=4\pi$. In that way, the statistical factor depends only on the detuning Δ_2. In the resonant case ($\Delta_2=0$), we have $\alpha=1$ and the system obeys the Fermi statistics. On the other hand, in the far detuned case ($\Delta_2 \gg 1$) the statistical factor tends to zero ($\alpha\to 0$) and the system obeys the Bose statistics. In this particular situation, varying the detuning Δ_2 we can transmute continually the statistics of the system from Fermi to Bose, going through the anyonic statistics. The statistical factor α depends on Δ_2 with the same pattern showed in Fig. 1(a) for $m=2$ (dashed line). We note that for the case $m=1$ the value $\alpha=1$ cannot be reached. In this latter case, the statistical factor can vary continually from 0 to $\frac{1}{2}$ depending on the detuning Δ_1 for the fixed solid angle $\Omega=4\pi$.

The scenario presented here for the simulation of one anyon statistics can be generalized for two anyons. Since the system composed by a fermion interacting with two bosonic fields behaves like one anyon, the generalization to two anyons can be achieved by adding two more bosonic fields (c,d). Let us suppose that the dynamics of the system is governed by the parametrized Hamiltonian (in the interaction picture) $\tilde{\mathbf{H}}(\theta,\phi)=\lambda_m\tilde{U}(\theta,\phi)\ [\sigma_+(a)^m(c)^m+\sigma_-(a^\dagger)^m(c^\dagger)^m]\ \tilde{U}^\dagger(\theta,\phi)$, under the unitary transformation $\tilde{U}(\theta,\phi) =\exp[-i\phi(J_z^{ab}+J_z^{cd})]\exp[-i\theta(J_y^{ab}+J_y^{cd})]$ (where J_z^{lk} and J_y^{lk} are the Schwinger angular momentum operators for modes l and k). Under cyclic and suitably slow variation of parameters θ and ϕ, the eigenstates of this new Hamiltonian, given by $|\tilde{\Psi}_0^{\pm}\rangle=(|\uparrow\rangle|0\rangle_a|0\rangle_c\pm|\downarrow\rangle|m\rangle_a|m\rangle_c)|0\rangle_b|0\rangle_d/\sqrt{2}$, evolve adiabatically. If θ and ϕ are eventually brought back to their initial values, $|\tilde{\Psi}_0^{\pm}\rangle$ acquires the Berry phase $\tilde{\gamma}_0=(m/2)\Omega$, which corresponds to twice the phase obtained in the previous case. Consequently, for the two-anyon case we have the statistical factor $\alpha=(m/2)(\Omega/2\pi)$, which is exactly twice as much as the one described in Eq. (6) for $\Delta_m=0$.

Finally, we discuss how to implement, in a physical context, Hamiltonian (1) and the unitary transformation (4). To this end we consider one single ion in a two-dimensional harmonic electromagnetic trap in the (x,y), plane, with degenerate frequency ν. The ion has two effective electronic states, $|\uparrow\rangle$ and $|\downarrow\rangle$, separated by the frequency ω_0 and coupled by the interaction with an effective laser plane wave propagating initially in the x direction, with frequency ω_L and wave vector $\vec{k}_L=(\omega_L/c)\vec{x}$. In this configuration, only the ionic motion along the x axis will be modified and the Hamiltonian of this system is given by [14]

$$H=\nu a^\dagger a+\nu b^\dagger b+\frac{\omega_0}{2}\sigma_z+g[\sigma_+e^{i\vec{k}_L\cdot\vec{r}-i\omega_L t+i\varphi_L}+\text{H.c.}], \quad (7)$$

where $\vec{r}$ is the vibration direction of the ion, g is the effective coupling constant for transition $|\uparrow\rangle\leftrightarrow|\downarrow\rangle$, and φ_L is the phase of the laser [15]. The effective laser beam is tuned to the mth red vibrational sideband of the ion, i.e., it is detuned by $\delta=\omega_0-\omega_L=m\nu$ from the $|\uparrow\rangle\leftrightarrow|\downarrow\rangle$ transition. In the interaction picture and in the frame rotating at the effective laser frequency ω_L, the Hamiltonian of this system can be written as [14,16]

$$\mathbf{H}=f_m(\eta,a^\dagger a)e^{i\varphi_L}\sigma_+(a)^m+\text{H.c.}, \quad (8)$$

where

$$f_m(\eta,a^\dagger a)=\frac{1}{2}ge^{-\eta^2/2}\left[\sum_{l=0}^{\infty}\frac{(i\eta)^{2l+m}}{l!\,(l+m)!}(a^\dagger)^l(a)^l+\text{H.c.}\right]$$

is the effective coupling, $\eta=\sqrt{(\vec{k}_L\cdot\vec{r})^2/2\mathcal{M}\nu}$ is the Lamb-Dicke parameter, and $\mathcal{M}$ is the ion mass. When we assume the so-called Lamb-Dicke regime $\eta\ll 1$, we have $f_m(\eta,a^\dagger a)\sim[1/2(m!)]g(i\eta)^m e^{-\eta^2/2}\equiv\lambda_m$. Different values of the nonlinear interaction order m can be reached by the choice of the mth red vibrational sideband of the ion. In this way, we can implement Hamiltonian (1). If we consider the effective pumping laser beam tuned not exactly on the mth vibrational sideband of the ion, we can obtain the off-resonance case of Eq. (1) with a small detuning limited by the frequencies of the neighboring side bands ($\Delta_m \ll \nu$). In this context, the statistical transmutation of the system can be investigated varying the solid angle Ω that depends on the path choice for parameters θ,ϕ. The unitary transformation (4) can be implemented by (i) the physical rotation, in the x,y plane, of the propagation direction of the effective laser field, in a way that $\vec{k}_L=(\omega_L/c)[\cos(\theta/2)\vec{x}+\sin(\theta/2)\vec{y}]$, together with (ii) a suitable variation on phase φ_L (note that $\varphi_L=2\varphi$). Therefore, by the control of the propagation direction of the laser and its phase, it is possible to implement physically the unitary transformation (4).

The fractional phase acquired by the ion due to unitary transformation (4) can be measured by a Ramsey-type interferometer similar to the one suggested in [11]. Let us start with the assumption that the vibrational modes of the ion in both directions x,y are cooled to their ground states, and its electronic internal levels are prepared in a superposition $(|\uparrow\rangle+|\downarrow\rangle)|0\rangle_a|0\rangle_b/\sqrt{2}$. This superposition can be generated by a carrier-type laser pulse [14], which corresponds to the choice $m=0$ (the laser is tuned in the $|\uparrow\rangle\leftrightarrow|\downarrow\rangle$ transition frequency, $\omega_0=\omega_L$), and does not affect the vibrational modes of the ion. The next step consists of turning on a laser tuned to the mth red vibrational sideband and initially aligned in the x direction with reference phase $\varphi_L=0$. This laser interacts with the ion for a time $\tau\gg 1/\lambda_m$, during which its propagation direction is rotated and φ_L is cyclically changed. The variation rate of these parameters must be much smaller than the effective coupling λ_m to allow the adiabatic regime assumed in our approach. When the laser is turned off, the system has evolved to state $(e^{i\gamma_{0,0}}|\uparrow\rangle+|\downarrow\rangle)|0\rangle_a|0\rangle_b/\sqrt{2}$; here we have used the fact that both eigenstates $|\Psi^+_{n,n'}\rangle$ and $|\Psi^-_{n,n'}\rangle$ acquire the same geometrical phase $\gamma_{0,0}$ and $|\uparrow\rangle|0\rangle_a|0\rangle_b=(|\Psi^+_{0,0}\rangle+|\Psi^-_{0,0}\rangle)/\sqrt{2}$. It is clear that the variation of parameters θ and φ_L must be designed such that the end of its cyclic path coincides with the end of the jth Rabi cycle performed in the time $\tau=2\pi j/\lambda_m (j\gg 1)$. Finally, we turn on again a carrier-type laser pulse ($m=0$) with phase $\varphi_L=\pi/2$ during time $t=\pi/(2g)$. After this rotation, we perform a fluorescence measurement of the electronic states of the ion [14], with probability $P_\downarrow=[1-\cos(\gamma_{0,0})]/2$ to measure level $|\downarrow\rangle$. In this way, we can measure the fractional phase introduced by the above procedure.

In this paper, we have presented a method to simulate the dynamics of anyons via the Jaynes-Cummings model. We have shown how to simulate anyons with $m/2,m/3,m/4,\ldots$ statistics or even how to transmute continually the statistics of the system from Fermi to Bose, going through the anyonic statistics. Such fractional features depend crucially on the entangled form of the system's eigenstate which works as a two-dimensional confinement. We also provide a proposal for the physical implementation of the above ideas in the context of trapped ions phenomena. It introduces a novel possibility for the investigation of fractional statistics in a well-controlled way.

This research was supported by the Engineering and Physical Sciences Research Council, the European Commission, and Elsag-spa company. M.F.S. and R.M.S. acknowledge the support of CNPq.

[1] F. Wilczek, Phys. Rev. Lett. **48**, 1144 (1982); **49**, 957 (1982).
[2] G. C. Canright and S. M. Girvin, Science **247**, 1197 (1990), and references cited therein.
[3] D. Arovas, J. R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. **53**, 722 (1984); B. I. Halperin, *ibid.* **52**, 1583 (1984).
[4] Z. F. Ezawa, *Quantum Hall Effects: Field Theoretical Approach and Related Topics* (World Scientific, Singapore, 2000).
[5] A. Y. Kitaev, e-print quant-ph/9707021; Ann. Phys. (N.Y.) **303**, 2 (2003).
[6] M. V. Berry, Proc. R. Soc. London, Ser. A **392**, 45 (1984).
[7] A. Shapere and F. Wilczek, *Geometric Phases in Physics* (World Scientific, Singapore, 1989).
[8] M. V. Berry and J. M. Robbins, Proc. R. Soc. London, Ser. A **453**, 1771 (1997).
[9] W. Vogel, D.-G. Welsch, and L. Leine, J. Opt. Soc. Am. B **4**, 1633 (1987); W. Vogel and D.-G. Welsch, Phys. Rev. A **40**, 7113 (1989).
[10] I. Fuentes-Guridi, A. Carollo, S. Bose, and V. Vedral, Phys. Rev. Lett. **89**, 220404 (2002).
[11] A. Carollo, M. França Santos, and V. Vedral, Phys. Rev. A **67**, 063804 (2003).
[12] B. Paredes, P. Fedichev, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **87**, 010402 (2001).
[13] S. Forte, Rev. Mod. Phys. **64**, 193 (1982).
[14] D. J. Wineland *et al.*, J. Res. Natl. Inst. Stand. Technol. **103**, 259 (1998); C. J. Myatt *et al.*, Nature (London) **403**, 269 (2000).
[15] Ch. Roos *et al.*, Phys. Rev. Lett. **83**, 4713 (1999).
[16] W. Vogel and R. L. de Matos Filho, Phys. Rev. A **52**, 4214 (1995).